Chapter 7 — LINE INTEGRALS AND GREEN'S THEOREM

The basic idea of analysis is the suitable approximation of complicated functions by simpler ones, such as linear functions. Thus a differentiable function will be one that is, near every point in its domain of definition, approximable by a linear function. It is our purpose to discover what knowledge about the function is deducible from knowledge of this approximation, called its differential. Two hundred years ago it might have been said that the differential expresses the infinitesimal, or instantaneous behavior of the function and the total behavior is the sum of its infinitesimal parts. Nowadays, it is generally conceded that such an assertion is nonsense; nevertheless it serves to describe the mood of the analyst as he begins his investigations.

Up until now we have been mainly concerned with one-dimensional calculus; although some of the applications have led us into the plane and space, our techniques have been mainly one dimensional. In the present chapter we turn to two dimensions, and in the next chapter we shall deal with the calculus of three dimensions. Each dimension has its own flavor. In one dimension, the order of the real numbers plays an important role; in two, we have the influence of complex numbers; and in three, we discover the vector product. However, there is also much that is the same in all these dimensions, and for these common concepts there is much to be gained from a unified treatment. Thus we begin the present chapter with a study of differentiable R^m-valued functions of n variables. We will be interested in mappings from R^1 to R^2, R^3 to R^2, and so on, but the concept of differentiability is the *same* in all cases and it is important for us to take cognizance of that fact. An R^m-valued function $\mathbf{f}$ defined in a neighborhood of a point $\mathbf{p}$ in R^n will be said to be differentiable at $\mathbf{p}$ if it can be suitably approximated near $\mathbf{p}$ by a linear transformation of R^n to R^m. This definition will make precise our usage up to now of the word differentiable. The transformation, whose existence is required, is called the differential of $\mathbf{f}$ and is denoted $d\mathbf{f}(\mathbf{p})$. We shall see that a differentiable R^m-valued function is an m-tuple of differentiable real-valued functions. We have already studied such functions in R^2, where we showed that if a function f has continuous first partial derivatives near $\mathbf{p}$, then it is differentiable there, and the differential is given by

$$df(\mathbf{p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, dx^i(\mathbf{p})$$

where $x^1, \ldots, x^n$ are the rectangular coordinate functions of R^n.

We have studied, in Chapter 1, some examples of coordinate systems for R^2 and R^3. We shall want, in the subsequent chapters, to consider more general kinds of coordinates. A coordinate system near a point $\mathbf{p}$ in R^n arises in this way: if $\mathbf{F}$ is a continuously differentiable R^n-valued function defined near $\mathbf{p}$, and the differential $d\mathbf{F}(\mathbf{p})$ is a nonsingular linear transformation, then the functions

$$y^1 = F^1(x), \ldots, y^n = F^n(x)$$

are coordinates in a neighborhood of $\mathbf{p}$. That is, the values of $y^1, \ldots, y^n$ serve to identify all points near $\mathbf{p}$. This fact, that the nonsingularity of the differential implies that of the mapping, is called the inverse mapping theorem. It asserts that the mapping $\mathbf{F}$ has an inverse near $\mathbf{p}$ when its differential at $\mathbf{p}$ does.

Suppose that f is a differentiable real-valued function defined in a domain D. Then its differential associates to each point in D a linear function on R^n. Any rule which does this is called a differential form. An important question which we shall study in this: *just when is a differential form the differential of a function*? In one variable, this question is easily answered. For if f is a differentiable function of a real variable, its differential is given by

$$f'(x)\, dx$$

Any continuous differential form in one variable is of the form $g(x)\, dx$. We know from the fundamental theorem of calculus that if G is an indefinite integral of g:

$$G(x) = \int_a^x g(t)\,dt$$

then G is differentiable and $dG = g\,dx$. Thus the answer to our question in one variable is *always*. The situation in several variables is not so easy. But the extension of the idea of integration to differential forms provides us with a tool for answering this question, and a several variable analog of the fundamental theorem (Green's theorem in R^2).

Green's theorem provides us with a tool to extensively study complex differentiable functions. This is the **Cauchy integral formula** which gives a means for determining such a function at interior points of a domain by its boundary values. It follows easily from this formula (a generalization of the formula given in Section 6.7) that a complex differentiable function must be **analytic**: expressible as a convergent power series. In fact, the entire behavior of such functions can be read off from the integral formula; this is the basis of the Cauchy theory of complex variables. We shall only begin this study.

7.1 The Differential

In Chapter 2 we studied differentiation of real-valued functions of many variables, differentiating with respect to one variable at a time. This gave us the concept of partial derivatives which generalized to the direction derivatives $df(\mathbf{p}, \mathbf{v})$ of a function f at a point $\mathbf{p}$ and in a direction $\mathbf{v}$. According to Proposition 20 of Chapter 2 if the partial derivatives are continuous in a neighborhood of $\mathbf{p}$, then the directional derivative $df(\mathbf{p}, \mathbf{v})$ varies linearly in $\mathbf{v}$. This linear function we called the differential of f at p. Now we shall give a more precise definition of this notion, in a style more like the definition of the derivative of an R^m-valued function of a real variable (see Proposition 5 of Chapter 3).

Definition 1. Let $\mathbf{p} \in R^n$, and suppose $\mathbf{f}$ is an R^m-valued function defined on a neighborhood of $\mathbf{p}$. We say that $\mathbf{f}$ is **differentiable at p** if there is a linear transformation $T\colon R^n \to R^m$ and a nonnegative real-valued function ε of a real variable such that $\lim\limits_{t \to 0} \varepsilon(t) = 0$ and

$$\|\mathbf{f}(\mathbf{p} + \mathbf{v}) - \mathbf{f}(\mathbf{p}) - T(\mathbf{v})\| \leq \varepsilon(\|\mathbf{v}\|)\|\mathbf{v}\| \tag{7.1}$$

when $\|\mathbf{v}\|$ is sufficiently small.

If such a linear transformation exists it is called the **differential of f at p** and is denoted by $d\mathbf{f}(\mathbf{p})$.

Notice that there can be at most one linear transformation T satisfying these requirements. For suppose also $S: R^n \to R^m$ satisfies (7.1). Then

$$\|S(\mathbf{h}) - T(\mathbf{h})\| \leq 2\varepsilon(\|\mathbf{h}\|)\|\mathbf{h}\|$$

for sufficiently small $\mathbf{h}$. Let $\mathbf{h} = t\mathbf{v}$ and take the limit as $t \to 0$,

$$\|S(t\mathbf{v}) - T(t\mathbf{v})\| = |t| \, \|S(\mathbf{v}) - T(\mathbf{v})\| \leq 2\varepsilon(t)|t| \, \|\mathbf{v}\|$$

thus $\|S(\mathbf{v}) - T(\mathbf{v})\| \leq 2\varepsilon(t)\|\mathbf{v}\|$ for all small t. Letting $t \to 0$, we obtain $S(\mathbf{v}) = T(\mathbf{v})$. Thus $S = T$.

Examples

1. $f(x, y) = xy^2$ is differentiable in the plane. Let $(x_0, y_0) \in R^2$ and let (h, k) be any vector. Then

$$\begin{aligned} f(x_0 + h, y_0 + k) = (x_0 + h)(y_0 + k)^2 &= x_0 y_0^2 + y_0^2 h + 2y_0 hk \\ &\qquad + 2x_0 y_0 k \\ &= x_0 y_0^2 + y_0^2 h + 2x_0 y_0 k + 2y_0 hk + x_0 k^2 + hk^2 \end{aligned}$$

Thus

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) - (y_0^2 h + 2x_0 y_0 k) = 2y_0 hk + x_0 k^2 + hk^2$$

This in norm is dominated by

$$\begin{aligned} &2|y_0|\,|hk| + |x_0|\,|k|^2 + |h|\,|k|^2 \\ &\quad \leq 2|y_0|(h^2 + k^2) + |x_0|(h^2 + k^2) + |h|(h^2 + k^2) \\ &\quad \leq \|(h, k)\|[\|(h, k)\|(|y_0| + |x_0|) + \|(h, k)\|] \end{aligned}$$

since $\|(h, k)\| = (h^2 + k^2)^{1/2}$.

Thus xy^2 is differentiable and has the differential at (x_0, y_0):

$$(h, k) \to y_0^2 h + 2x_0 y_0 k$$

This means that for small values of (h, k), the difference

$$(x_0 + h)(y_0 + k)^2 - x_0 y_0{}^2$$

is effectively approximable by

$$y_0{}^2 h + 2x_0 y_0 k$$

The meaning of "effective" is that the error in this approximation is of the order of $\varepsilon \|(h, k)\|$, where ε can be made as small as we please, by choosing the neighborhood of (x_0, y_0) small enough.

2. More generally, Proposition 20 of Chapter 2 suggests that a real-valued function with continuous partial derivatives near $\mathbf{p}_0$ is differentiable there. This means that for small values of $\mathbf{v}$, $f(\mathbf{p}_0 + \mathbf{v}) - f(\mathbf{p}_0)$ is effectively approximable by $\langle \nabla f(\mathbf{p}_0), \mathbf{v} \rangle = (\sum \partial f/\partial x^i(\mathbf{p}_0)v^i)$. Let us complete Proposition 20 of Chapter 2 to a verification of this fact (at least in R^2). By the mean value theorem we may write, for $\mathbf{p} = (x_0, y_0)$, $\mathbf{v} = (h, k)$:

$$f(\mathbf{p} + t\mathbf{v}) - f(\mathbf{p}) = \frac{\partial f}{\partial x}(\xi_0, y_0)h + \frac{\partial f}{\partial y}(x_0 + th, \eta_0)k$$

where $|\xi_0 - x_0| \le h$, $|\eta_0 - y_0| \le k$. Then

$$\left| f(\mathbf{p} + t\mathbf{v}) - f(\mathbf{p}) - \frac{\partial f}{\partial x}(\mathbf{p})h + \frac{\partial f}{\partial y}(\mathbf{p})k \right|$$
$$\le \left| \left[\frac{\partial f}{\partial x}(\mathbf{p}_1) - \frac{\partial f}{\partial x}(\mathbf{p})\right] h + \left[\frac{\partial f}{\partial y}(\mathbf{p}_2) - \frac{\partial f}{\partial x}(\mathbf{p})\right] k \right| \tag{7.2}$$

where $\mathbf{p}_1, \mathbf{p}_2$ are at least as close to $\mathbf{p}$ as $\mathbf{p} + \mathbf{v}$. By Schwarz's inequality (7.2) is dominated by

$$\left\| \left(\frac{\partial f}{\partial x}(\mathbf{p}_1) - \frac{\partial f}{\partial x}(\mathbf{p}), \frac{\partial f}{\partial y}(\mathbf{p}_2) - \frac{\partial f}{\partial y}(\mathbf{p}) \right) \right\| \|\mathbf{v}\|$$

and the first term is dominated by

$$\varepsilon(\|\mathbf{v}\|) = \max\left\{ \left\| \left(\frac{\partial f}{\partial x}(\mathbf{p}_1) - \frac{\partial f}{\partial x}(\mathbf{p}), \frac{\partial f}{\partial y}(\mathbf{p}_2) - \frac{\partial f}{\partial y}(\mathbf{p}) \right) \right\| \right\}$$

all $\mathbf{p}_1$, $\mathbf{p}_2$ in the ball $B(\mathbf{p}, \|\mathbf{v}\|)$

which tends to zero $\|\mathbf{v}\| \to 0$.

3. **Error analysis.** The differential of a function gives us approximately the difference between two values of a function in terms of the difference between the variables:

$$f(\mathbf{x}) - f(\mathbf{x}_0) = \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{x}_0 \rangle + \text{error} \tag{7.3}$$

where the error is negligible if the difference is small. Considered this way, the differential may be used to compute tolerance levels for errors in measurement. For example, we can compute the maximal error in the volume of a rectangular box, given certain tolerances in the measurements of the sides. Suppose the sides can be measured within an error of 2%. The function we are concerned with is $f(x, y, z) = xyz$ and $\nabla f = (yz, xz, xy)$. The error in the measurement of a volume will be, according to (7.3), approximately equal to

$$\langle (yz, xz, xy), 0.02(x, y, z) \rangle = 3[0.02(xyz)]$$

Thus, the percentage error is

$$100 \frac{f(x) - f(x_0)}{f(x_0)} = 100 \frac{0.06(xyz)}{xyz} = 6\%$$

Thus an error is magnified threefold.

4. Let $f(x, y, z) = x(\cos y)e^{x+z}$. Given error tolerances of 2%, 1%, 5% in the measurements of x, y, z, respectively, what error is possible in the computation of f?

Here

$$\nabla f = ((\cos y)e^{x+z}(1 + x), -x(\sin y)e^{x+z}, x(\cos y)e^{x+z})$$

The ratio of the increment in f to the computed value of f is approximately

$$\left| \frac{\nabla f(x, y, z), (0.02x, 0.02y, 0.02z)}{f(x, y, z)} \right|$$
$$= (1 + x)(0.02) + y(\tan y)(0.01) + (0.05)z$$

Here we see that the error in the computed value of f depends on the magnitude of the variables. If y is close to $\pi/2$, the error is very bad. The maximum percent error for values of x, y, z in these ranges: $|x| \leq 1$, $|y| \leq \pi/4$, $|z| \leq 1$, is

$$2(2) + \frac{\pi}{4}(1) + 5(1) = 9 + \frac{\pi}{4}$$

which is less than 10.

5. A linear transformation is differentiable at every point. Let $T: R^n \to R^m$ be a given linear transformation, and let $\mathbf{p} \in R^n$. Since

$$T(\mathbf{p} + \mathbf{v}) - T(\mathbf{p}) = T(\mathbf{v})$$

we have $\|T(\mathbf{p} + \mathbf{v}) - T(\mathbf{p}) - T(\mathbf{v})\| = 0$, so the estimate required by the definition is precise. Furthermore, for any $\mathbf{p} \in R^n$, $dT(\mathbf{p}) = T$.

In particular, the coordinate functions $x^1, \ldots, x^n$ are differentiable and $dx^i(\mathbf{p}, \mathbf{v}) = v^i$ for any $\mathbf{p}$, $\mathbf{v}$. Since dx^i is independent of the base point we shall often omit it. Notice, that $dx^1, \ldots, dx^n$ form a basis for the space of linear functions on R^n, so the differential of any function will be a linear combination of these differentials. In particular, if f is differentiable at $\mathbf{p}$, we have

$$df(\mathbf{p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, dx^i \tag{7.4}$$

We have just shown that in two dimensions, but it is easier to directly compare Definition 1 of this chapter and Definition 14 of Chapter 2 (cf. Problem 1) to obtain

$$df(\mathbf{p})(\mathbf{E}_i) = \lim_{t \to 0} f \frac{(\mathbf{p} + t\mathbf{E}_i) - f(\mathbf{p})}{t} = \frac{\partial f}{\partial x^i}(\mathbf{p})$$

The verification of the following proposition concerning the behavior of the differential under algebraic operations are easily performed.

Proposition 1.
(i) *Suppose that* $\mathbf{f}, \mathbf{g}$ *are differentiable* R^m*-valued functions at* $\mathbf{p}$. *Then* $\mathbf{f} + \mathbf{g}$ *and* $\langle \mathbf{f}, \mathbf{g} \rangle$ *are also differentiable and*

$$d(\mathbf{f} + \mathbf{g})(\mathbf{p}) = d\mathbf{f}(\mathbf{p}) + d\mathbf{g}(\mathbf{p})$$
$$d\langle \mathbf{f}, \mathbf{g} \rangle(\mathbf{p}) = \langle d\mathbf{f}(\mathbf{p}), \mathbf{g}(\mathbf{p}) \rangle + \langle \mathbf{f}(\mathbf{p}), d\mathbf{f}(\mathbf{p}) \rangle$$

(ii) *Suppose* $\mathbf{f} = (f^1, \ldots, f^m)$ *is an* R^m*-valued function defined in a neighborhood of* $\mathbf{p}$. $\mathbf{f}$ *is differentiable at* $\mathbf{p}$ *if and only if* $f^1, \ldots, f^m$ *are. In this case we have*

$$d\mathbf{f}(\mathbf{p}) = (df^1(\mathbf{p}), \ldots, df^m(\mathbf{p}))$$

Proof. We shall only verify the differentiability of $\langle \mathbf{f}, \mathbf{g} \rangle$; the other assertions are clear. By the hypothesis of (i) there are functions ε, η of a real variable such that $\lim \varepsilon(t) = \lim (t) = 0$ as $t \to 0$, and linear transformations R, S such that

$$\|\mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p}) - R(\mathbf{v})\| \leq \varepsilon(\|\mathbf{v}\|)\,\|\mathbf{v}\| \tag{7.5}$$

$$\|\mathbf{g}(\mathbf{p}+\mathbf{v}) - \mathbf{g}(\mathbf{p}) - S(\mathbf{v})\| \leq \eta(\|\mathbf{v}\|)\,\|\mathbf{v}\| \tag{7.6}$$

Let $h(\mathbf{x}) = \langle \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \rangle$. Then

$$\begin{aligned} h(\mathbf{p}+\mathbf{v}) - h(\mathbf{p}) &= \langle \mathbf{f}(\mathbf{p}+\mathbf{v}), \mathbf{g}(\mathbf{p}+\mathbf{v}) \rangle - \langle \mathbf{f}(\mathbf{p}), \mathbf{g}(\mathbf{p}) \rangle \\ &= \langle \mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p}), \mathbf{g}(\mathbf{p}+\mathbf{v}) \rangle \\ &\quad + \langle \mathbf{f}(\mathbf{p}), \mathbf{g}(\mathbf{p}+\mathbf{v}) - \mathbf{g}(\mathbf{p}) \rangle \end{aligned} \tag{7.7}$$

If we replace the first term by $R(\mathbf{v})$ we commit an error of $\varepsilon(\|\mathbf{v}\|)\,\|\mathbf{v}\|\,\|\mathbf{g}(\mathbf{p}+\mathbf{v})\|$ and if we replace the last term by $S(\mathbf{v})$ we commit an error of $\eta(\|\mathbf{v}\|)\,\|\mathbf{v}\|\,\|\mathbf{f}(\mathbf{p})\|$. These are admissible errors, so we shall bravely proceed with these replacements. From (7.7), we obtain

$$\begin{aligned} &|h(\mathbf{p}+\mathbf{v}) - h(\mathbf{p}) - (\langle R(\mathbf{v}), \mathbf{g}(\mathbf{p}) \rangle + \langle \mathbf{f}(\mathbf{p}), S(\mathbf{v}) \rangle)| \\ &\quad \leq |\langle \mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p}) - R(\mathbf{v}), \mathbf{g}(\mathbf{p}+\mathbf{v}) \rangle| + |\langle R(\mathbf{v}), \mathbf{g}(\mathbf{p}+\mathbf{v}) - \mathbf{g}(\mathbf{p}) \rangle| \\ &\quad + |\langle \mathbf{f}(\mathbf{p}), \mathbf{g}(\mathbf{p}+\mathbf{v}) - \mathbf{g}(\mathbf{p}) - S(\mathbf{v}) \rangle| \\ &\quad \leq \varepsilon(\|\mathbf{v}\|)\,\|\mathbf{v}\|\,\|\mathbf{g}(\mathbf{p}+\mathbf{v})\| + \|R(\mathbf{v})\|(\|S(\mathbf{v})\| + \eta(\|\mathbf{v}\|)\,\|\mathbf{v}\|) \\ &\quad + \|\mathbf{f}(\mathbf{p})\|\eta(\|\mathbf{v}\|)\,\|\mathbf{v}\| \end{aligned}$$

If we take M larger than the maximum value of $\|\mathbf{g}(\mathbf{p}+\mathbf{v})\|$, and also larger than $\|R\|$ and $\|S\|$, this is dominated by

$$[M\varepsilon(\|\mathbf{v}\|) + M^2\|\mathbf{v}\| + M\,\|\mathbf{v}\|\,\eta(\|\mathbf{v}\|) + \|\mathbf{f}(\mathbf{p})\|\,\eta(\|\mathbf{v}\|)]\,\|\mathbf{v}\|$$

which is of the desired form.

Examples

6. $f(x, y) = e^x \cos y + y^x$.
$df(x, y) = (e^x \cos y + y^x \log y)\,dx + (-e^x \sin y + xy^{x-1})\,dy$

7. $f(x, y, z) = xyz$, $df(x, y, z) = yz\,dx + xz\,dy + xy\,dz$.

8. $f(x, y, z) = \begin{pmatrix} z & z^2 \\ 0 & z \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

$$df(x, y, z) = \begin{pmatrix} z & z^2 \\ 0 & z \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} + \begin{pmatrix} dz & 2z\,dz \\ 0 & dz \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

• EXERCISES

1. Find the differential of these functions:
   (a) $y \cos x + \sin zx$.
   (b) $\cos(e^{x+y}) + \cos(xe^y)$.
   (c) $\exp\langle \mathbf{x}, \mathbf{a} \rangle$.
   (d) $\langle \mathbf{x}, \exp\langle \mathbf{x}, \mathbf{a} \rangle \rangle$.
   (e) $x^2 + y^2 + zx$.
   (f) $(x - y)e^{x+y}$.
   (g) $\Pi_{i=1}^n x^i$.
2. For each of the following functions, in how large an interval about the origin may we estimate $f(\mathbf{v}) - f(\mathbf{0})$ by $\langle \nabla f(\mathbf{0}), \mathbf{v} \rangle$ incurring an error of at most $10^{-3} \|\mathbf{v}\|$?
   (a) xy (d) $\sin(x + 2y)$
   (b) e^{x+y} (e) $x + e^{2y}$
   (c) $\sin x + \cos y$ (f) $\exp(x^2 + y^2)$
3. In how large a disk about the point $\mathbf{p} \neq 0$ can we estimate the polar coordinates of nearby points $\mathbf{p} + \mathbf{v}$ by a linear function, with an error of at most $10^{-3} \|\mathbf{v}\|$?

• PROBLEMS

1. Suppose that f is a differentiable real-valued function defined in a neighborhood of $\mathbf{p}$ in R^n. Using the definition, verify that

$$df(\mathbf{p})(\mathbf{E}_i) = \lim_{t \to 0} \frac{f(\mathbf{p} + t\mathbf{E}_i) - f(\mathbf{p})}{t} = \frac{\partial f}{\partial x_i}(\mathbf{p})$$

and conclude that

$$df(\mathbf{p}) = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i}\right)(\mathbf{p})\, dx^i$$

2. Let $\mathbf{M}(\mathbf{x})$, $\mathbf{N}(\mathbf{x})$ be $n \times n$ matrix valued functions of the variable $\mathbf{x}$. If $\mathbf{M}$, $\mathbf{N}$ are differentiable at $\mathbf{p}$, so is $\mathbf{MN}$. Show that $d(\mathbf{MN})(\mathbf{p}) = d\mathbf{M}(\mathbf{p})\mathbf{N} + \mathbf{M}\, d\mathbf{N}(\mathbf{p})$.
3. If $f(t) = \det(\exp(\mathbf{M}t))$, show that $f'(0) = tr\mathbf{M}$.

4. A quantity Q varies with x, y, z according to

$$Q = \frac{e^x}{yz}$$

Suppose that x, y, z can be measured to within an error of 1%, 1/2%, 3%, respectively. What will be the corresponding maximal error in Q at corresponding values? At (a) $x = 0, y = 2, z = 5$; (b) $x = 2, y = 1, z = 3$, in particular?

7.2 Coordinate Changes

In Chapter 1 we introduced some systems of coordinates in R^n, and we saw that for certain problems a change of coordinates made the problem understandable and solvable. Later on we saw, in the study of systems of linear differential equations, that it was convenient, where possible, to switch to coordinates relative to a basis of eigenvectors. In the geometric study of surfaces, and in many physical problems it is advantageous to admit very general coordinate changes. We now introduce a general notion of coordinates.

Definition 2. Let U be a domain in R^n. A **system of coordinates** is an n-tuple of continuously differentiable functions $\mathbf{y} = (y^1, \ldots, y^n)$ defined on U such that

(i) if $\mathbf{p} \neq \mathbf{q}$, then $\mathbf{y}(\mathbf{p}) \neq \mathbf{y}(\mathbf{q})$,
(ii) $dy^1(\mathbf{p}), \ldots, dy^n(\mathbf{p})$ are independent at all $\mathbf{p} \in U$.

The first condition states that any point is uniquely determined by the value of $\mathbf{y}$ at that point. In this sense $y^1, \ldots, y^n$ are coordinates. We can name points in U by means of the functions $y^1, \ldots, y^n$. Further, if f is a function defined on U, we can describe it as a function of the coordinates $y^1, \ldots, y^n$. The second condition asserts that the differentials $dy^1, \ldots, dy^n$ span the space of linear functions. Thus we can express the differential of a function as a linear combination of these differentials; it should be no surprise that (7.4) is valid in any coordinate system.

Proposition 2. *Suppose that $y^1, \ldots, y^n$ are coordinates in a neighborhood of p. If f is a differentiable function defined in a neighborhood of* $\mathbf{p}$, *then*

$$df(\mathbf{p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial y^i}(\mathbf{p})\, dy^i(\mathbf{p})$$

Proof. Let $x^1, \ldots, x^n$ be the coordinates of R^n relative to the standard basis. We know that

$$df(\mathbf{p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, dx^i$$

Now we can express the standard coordinates as differentiable functions of the new coordinates $y^1, \ldots, y^n$: $x^i = x^i(y^1, \ldots, y^n)$, $i = 1, \ldots, n$, and f can be expressed as a function of $y^1, \ldots, y^n$ by composition:

$$f(\mathbf{p}) = f(x^1(\mathbf{y}(\mathbf{p})), \ldots, x^n(\mathbf{y}(\mathbf{p})))$$

Let us assume that $\mathbf{p}$ is the origin relative to both $\mathbf{x}$ and $\mathbf{y}$ coordinates. Now $\partial f/\partial y^i$ is the derivative of f with respect to y^i, holding the other variables y^j, $j \neq i$ constant. In other words, $\partial f/\partial y^i(\mathbf{p})$ is the derivative of f at $\mathbf{p}$ along the curve $y^j = 0$, $j \neq i$. We can parametrize this curve by

$$\begin{aligned} x^1 &= g^1(t) = x^1(0, \ldots, 0, t, 0, \ldots, 0) \\ &\;\;\cdots \\ x^n &= g^n(t) = x^n(0, \ldots, 0, t, 0, \ldots, 0) \end{aligned}$$

for t near 0. Now by Proposition 3 of Chapter 3, we have

$$\frac{\partial f}{\partial y^i}(\mathbf{p}) = \frac{d}{dt} f(g^1(t), \ldots, g^n(t))|_{t=0} = \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(0) \frac{dg^k}{dt}(0)$$

But $dg^k/dt(0) = \partial x^k/\partial y^i(0)$. Thus

$$\frac{\partial f}{\partial y^i}(\mathbf{p}) = \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(0) \frac{\partial x^k}{\partial y^i}(0) \tag{7.8}$$

As the x^i are differentiable functions of $\mathbf{y}$; $dx^i = \sum (\partial x^i/\partial y^j)\, dy^j$ and we conclude that

$$df(\mathbf{p}) = \sum_i \frac{\partial f}{\partial x^i}(\mathbf{p})\, dx^i = \sum_{i,j} \frac{\partial f}{\partial x^i}(\mathbf{p}) \frac{\partial x^i}{\partial y^j}\, dy^j = \sum_j \frac{\partial f}{\partial y^j}\, dy^j$$

Examples

9. Polar coordinates: the change of coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

is valid in any disk not containing the origin. We have

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta, \; dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

so

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

If f is any differentiable function,

$$\frac{\partial f}{\partial r} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \theta} = r\left(-\sin\theta\,\frac{\partial f}{\partial x} + \cos\theta\,\frac{\partial f}{\partial y}\right)$$

10. Spherical coordinates:

$$\begin{aligned}
&x = r\cos\theta\cos\phi \qquad y = r\sin\theta\cos\phi \qquad z = r\sin\phi\\
&dx = \cos\theta\cos\phi\, dr - r\sin\theta\cos\phi\, d\theta - r\cos\theta\sin\phi\, d\phi\\
&dy = \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi\\
&dz = \sin\phi\, dr + r\cos\phi\, d\phi
\end{aligned}$$

If f is differentiable,

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}\\
&= \cos\theta\cos\phi\,\frac{\partial f}{\partial x} + \sin\theta\cos\theta\,\frac{\partial f}{\partial y} + \sin\phi\,\frac{\partial f}{\partial z}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta}\\
&= r\left(-\sin\theta\cos\phi\,\frac{\partial f}{\partial x} + \cos\theta\cos\phi\,\frac{\partial f}{\partial y}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial \phi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \phi}\\
&= r\left(-\cos\theta\sin\phi\,\frac{\partial f}{\partial x} - \sin\theta\sin\phi\,\frac{\partial f}{\partial y} + \cos\phi\,\frac{\partial f}{\partial z}\right)
\end{aligned}$$

11. Let $f(x, y, z) = e^x yz$. Find $\partial f/\partial r$, $\partial f/\partial \theta$:

$$\frac{\partial f}{\partial r} = (\cos\theta\cos\phi)e^x yz + (\sin\theta\cos\phi)e^x z + (\sin\phi)e^x y$$

$$= r\exp(r\cos\theta\cos\phi)(r\sin\theta\cos\theta\cos^2\phi\sin\phi$$

$$+ 2\sin\theta\cos\phi\sin\phi)$$

$$\frac{\partial f}{\partial \theta} = r[(-\sin\theta\cos\phi)e^x yz + (\cos\theta\cos\phi)e^x z]$$

$$= r^2\exp(r\cos\theta\cos\phi)[-r\sin^2\theta\cos^2\phi\sin\phi + \cos\theta\cos\phi\sin\phi]$$

12. Find $\partial f/\partial x$ if $f(r, \theta, \phi) = \phi^2$ in spherical coordinates. In order to solve this we have to write f explicitly as a function of the rectangular coordinates. Since $\phi = \arcsin(z/r)$,

$$\frac{\partial f}{\partial x} = 2\phi\frac{\partial \phi}{\partial x} = 2\arcsin\frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$\times\frac{\partial}{\partial x}\left(\arcsin\frac{z}{(x^2+y^2+z^2)^{1/2}}\right)$$

The Jacobian

In general, if

$$y^1 = f^1(x^1, \ldots, x^n)$$
$$\cdots$$
$$y^n = f^n(x^1, \ldots, x^n)$$

is a change of coordinates, we shall write this as $\mathbf{y} = \mathbf{F}(\mathbf{x})$. The differential $d\mathbf{F}(\mathbf{x}_0)$ is a nonsingular linear transformation on R^n. The matrix relative to $\mathbf{x}$ coordinates representing this transformation is referred to as the **Jacobian of the mapping** and denoted (when it is of value to make the coordinates explicit) by

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} = \left(\frac{\partial y^i}{\partial x^j}\right) \qquad i, j = 1, \ldots, n$$

According to Proposition 2

$$\frac{\partial y^i}{\partial y^j} = \sum_{k=1}^{n}\frac{\partial y^i}{\partial x^k}\frac{\partial x^k}{\partial y^j}$$

which is just the entry by entry form of the equation

$$\mathbf{I} = \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} \frac{\partial(x^1, \ldots, x^n)}{\partial(y^1, \ldots, y^n)}$$

Thus the matrices are inverse to each other as are the corresponding differentials:

$$d\mathbf{F}^{-1}(\mathbf{y}_0) = [d\mathbf{F}(\mathbf{x}_0)]^{-1} \qquad \text{if } \mathbf{y}_0 = \mathbf{F}(\mathbf{x}_0)$$

Example

13. Let

$$u = x + e^y$$

$$v = x \cos y$$

be a coordinate change in a domain in R^2. Then

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} 1 & e^y \\ \cos y & -x \sin y \end{pmatrix}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{-1}{e^y \cos y + x \sin y} \begin{pmatrix} -x \sin y & -e^y \\ -\cos y & 1 \end{pmatrix}$$

If $f(u, v) = u^2 + v^2$, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 2u + 2v \cos y = 2(x + e^y + x \cos^2 y)$$

If $g(x, y) = x^2 + y^2$, then

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} = 2\,\frac{x^2 \sin y + ye^y}{e^y \cos y + x \sin y}$$

These observations form special cases of the multivariable chain rule. We have already seen (Propositions 3.2, 3.3) other special cases. The general situation is this: the differential of a composed function (see Figure 7.1) is the composition of the differentials:

$$d(\mathbf{g} \circ \mathbf{f})(\mathbf{p}) = d\mathbf{g}(\mathbf{f}(\mathbf{p})) \circ d\mathbf{f}(\mathbf{p}) \tag{7.9}$$

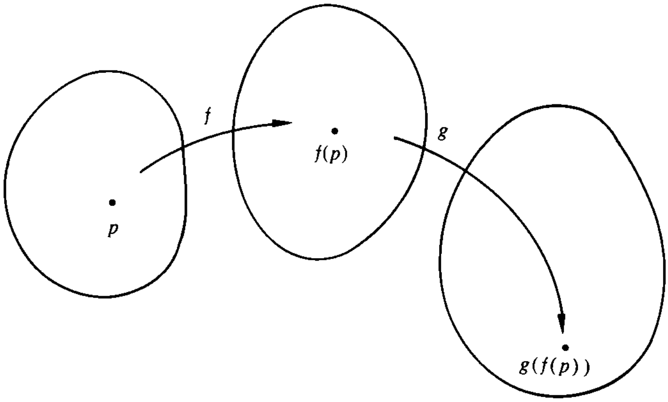


Figure 7.1

In coordinates this is easy to compute by linear algebra. Let $x^1, \ldots, x^n$ be coordinates in R^n, $y^1, \ldots, y^m$ in R^m and $z^1, \ldots, z^p$ in R^p. Then **f** and **g** are given in coordinates by

$$\mathbf{f}\colon y^i = f^i(x^1, \ldots, x^n) \qquad 1 \le i \le m$$

$$\mathbf{g}\colon z^j = g^j(y^1, \ldots, y^m) \qquad 1 \le i \le p$$

Let $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$. Then $\mathbf{h}$ is given by the p-tuple of functions

$$z^j = h^j(x^1, \ldots, x^m) = g^j(f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$$

(7.9) is the same as all these equations

$$\frac{\partial h^j}{\partial x^i}(\mathbf{p}) = \sum_{k=1}^{m} \frac{\partial g^j}{\partial y^k}(\mathbf{f}(\mathbf{p}))\, \frac{\partial f^k}{\partial x^i}(\mathbf{p}) \qquad 1 \le j \le p \quad 1 \le i \le n \tag{7.10}$$

This is true since $d\mathbf{g}(\mathbf{f}(\mathbf{p}))$, $d\mathbf{f}(\mathbf{p})$ are represented by the matrices

$$\left(\frac{\partial g^i}{\partial y^j}(f(\mathbf{p})\right), \left(\frac{\partial f^i}{\partial x^j}(\mathbf{p})\right)$$

respectively. We can rewrite (7.9) and (7.10) again in matrix form. The Jacobian of a product is the product of the Jacobians, and (7.9), (7.10) become

$$\frac{\partial(z^1, \ldots, z^p)}{\partial(x^1, \ldots, x^n)}(\mathbf{p}) = \frac{\partial(z^1, \ldots, z^p)}{\partial(y^1, \ldots, y^m)}(\mathbf{f}(\mathbf{p}))\frac{\partial(y^1, \ldots, y^m)}{\partial(x^1, \ldots, x^n)}(\mathbf{p})$$

Here is the proof of the chain rule.

Theorem 7.1. (The Chain Rule) *Let* $\mathbf{p}$ *be a point in* R^n. *Suppose* $\mathbf{f}$ *is a differentiable* R^m*-valued function defined in a neighborhood of* $\mathbf{p}$, *and* $\mathbf{g}$ *is a differentiable* R^p*-valued function defined in a neighborhood of* $\mathbf{f}(\mathbf{p})$. *Then* $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ *is differentiable at* $\mathbf{p}$ *and* $d\mathbf{h}(\mathbf{p}) = d\mathbf{g}(\mathbf{f}(\mathbf{p})) \circ d\mathbf{f}(\mathbf{p})$.

Proof. Let $T = d\mathbf{g}(\mathbf{f}(\mathbf{p}))$, and $S = d\mathbf{f}(\mathbf{p})$. We must show that

$$\|\mathbf{h}(\mathbf{p}+\mathbf{v}) - \mathbf{h}(\mathbf{p}) - T \circ S(\mathbf{v})\| \leq \varepsilon(\mathbf{v})\,\|\mathbf{v}\| \tag{7.11}$$

where $\lim\limits_{\|\mathbf{v}\| \to 0} \varepsilon(\mathbf{v}) = 0$. Let

$$\boldsymbol{\phi}(\mathbf{v}) = \mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p}) - S(\mathbf{v}) \tag{7.12}$$

$$\boldsymbol{\psi}(\mathbf{w}) = \mathbf{g}(\mathbf{f}(\mathbf{p}+\mathbf{w})) - \mathbf{g}(\mathbf{f}(\mathbf{p})) - T(\mathbf{w}) \tag{7.13}$$

Then, since $\mathbf{f}$, $\mathbf{g}$ are differentiable,

$$\|\boldsymbol{\phi}(\mathbf{v})\| \leq \delta(\mathbf{v})\,\|\mathbf{v}\| \qquad \|\boldsymbol{\psi}(\mathbf{w})\| \leq \eta(\mathbf{w})\,\|\mathbf{w}\|$$

where $\delta(\mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$ and $\eta(\mathbf{v}) \to 0$ as $\|\mathbf{w}\| \to 0$. Now, we verify (7.11) by computation:

$$\begin{aligned}\mathbf{h}(\mathbf{p}+\mathbf{v}) - \mathbf{h}(\mathbf{p}) &= \mathbf{g}(\mathbf{f}(\mathbf{p}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{p}))\\ &= \mathbf{g}(\mathbf{f}(\mathbf{p})) + T(\mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p})) + \boldsymbol{\psi}(\mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p})) - \mathbf{g}(\mathbf{f}(\mathbf{p}))\end{aligned}$$

(by taking $\mathbf{w} = \mathbf{f}(\mathbf{p}+\mathbf{v}) - \mathbf{f}(\mathbf{p})$ in (7.13)). Now using (7.12) we can continue:

$$\begin{aligned}&= T(S(\mathbf{v})) + \boldsymbol{\phi}(\mathbf{v})) + \boldsymbol{\psi}(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))\\ &= T \circ S(\mathbf{v}) + T(\boldsymbol{\phi}(\mathbf{v})) + \boldsymbol{\psi}(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))\end{aligned}$$

since T is linear. Thus

$$\|\mathbf{h}(\mathbf{p}+\mathbf{v}) - \mathbf{h}(\mathbf{p}) - T \circ S(\mathbf{v})\| \leq \frac{\|T(\boldsymbol{\phi}(\mathbf{v}))\| + \|\boldsymbol{\psi}(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))\|}{\|\mathbf{v}\|}\|\mathbf{v}\|$$

Now we must show that

$$\varepsilon(\mathbf{v}) = \frac{\|T(\boldsymbol{\phi}(\mathbf{v}))\|}{\|\mathbf{v}\|} + \frac{\|\boldsymbol{\psi}(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))\|}{\|\mathbf{v}\|} \to 0$$

as $\|\mathbf{v}\| \to 0$. As for the first term

$$\frac{\|T(\boldsymbol{\phi}(\mathbf{v})\|}{\|\mathbf{v}\|} \le \frac{\|T\| \, \|\boldsymbol{\phi}(\mathbf{v})\|}{\|\mathbf{v}\|} \le \|T\| \, \delta(\mathbf{v})$$

which tends to zero as $\mathbf{v} \to \mathbf{0}$, so that is alright. The second term is

$$\frac{\|\boldsymbol{\psi}(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))\|}{\|\mathbf{v}\|} \le \frac{\eta(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v})) \, \|S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v})\|}{\|\mathbf{v}\|}$$

$$\le \eta(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}))(\|S\| + \delta(\mathbf{v}))$$

As $\mathbf{v} \to \mathbf{0}$, so does $S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v}) \to 0$, and also $\eta(S(\mathbf{v}) + \boldsymbol{\phi}(\mathbf{v})) \to 0$. The final parenthesis is bounded so the whole term tends to zero. We are through.

Finally, we wish to give a sufficient condition that an n-tuple of functions $y^1 = f^1(x), \ldots, y^n = f^n(x)$ gives a coordinate system in a domain D in R^n. If $y^1, \ldots, y^n$ *are* coordinates, then we can invert these equations, that is, since the y's suffice to determine points in D, we can compute the x coordinates in terms of $y^1, \ldots, y^n$. Thus there are functions $x^1 = g^1(y), \ldots, x^n = g^n(y)$ such that

$$\mathbf{x} = \mathbf{g}(\mathbf{y}) \quad \text{if and only if} \quad \mathbf{y} = \mathbf{g}(\mathbf{x})$$

in the domain D. Now the second condition defining a coordinate system is that the differentials $df^1, \ldots, df^n$ are independent. The **inverse mapping theorem** asserts that if this second condition is valid at a point, then the first must hold in a neighborhood of that point. Thus the independence of the vectors $df^1(\mathbf{p}), \ldots, df^n(\mathbf{p})$ are enough to guarantee that $y^1, \ldots, y^n$ are coordinates near $\mathbf{p}$.

Theorem 7.2. (Inverse Function Theorem) *Let $\mathbf{F}$ be a continuously differentiable R^n-valued function defined in a neighborhood of $\mathbf{p}_0$ in R^n. Let $\mathbf{q}_0 = \mathbf{F}(\mathbf{p}_0)$. If the differential $d\mathbf{F}(\mathbf{p}_0)$ is nonsingular, then there is a neighborhood U of $\mathbf{q}$ and a continuously differentiable mapping $\mathbf{G}$ defined on U such that $\mathbf{G}(\mathbf{q}_0) = \mathbf{p}_0$ and for each $\mathbf{q}$ in U*

$$\mathbf{F}(\mathbf{p}) = \mathbf{q} \quad \textit{if and only if} \quad \mathbf{p} = \mathbf{G}(\mathbf{q})$$

Proof. Let us, for simplicity of notation, assume that $\mathbf{p}_0 = \mathbf{q}_0 = \mathbf{0}$. We have to show that if $\mathbf{q}$ is small enough the equation

$$\mathbf{F}(\mathbf{p}) - \mathbf{q} = \mathbf{0}$$

has a unique solution $\mathbf{p}$ in a neighborhood of $\mathbf{0}$. This suggests Newton's method for finding roots. The linear approximation to the mapping $\mathbf{p} \to \mathbf{F}(\mathbf{p}) - \mathbf{q}$ at a point $\mathbf{p}_1$ is given in terms of the differential:

$$\mathbf{p} \to \mathbf{F}(\mathbf{p}_1) - \mathbf{q} + d\mathbf{F}(\mathbf{p}_1)(\mathbf{p} - \mathbf{p}_1) \tag{7.14}$$

If $\mathbf{p}_1$ is near enough to 0, $d\mathbf{F}(\mathbf{p}_1)$ is nonsingular, so we can find a root of (7.14), namely,

$$\mathbf{p} = \mathbf{p}_1 - d\mathbf{F}(\mathbf{p}_1)^{-1}[\mathbf{F}(\mathbf{p}_1) - \mathbf{q}] \tag{7.15}$$

Now we consider the transformation T_q defined in a neighborhood of 0 by

$$T_q(\mathbf{p}) = \mathbf{p} - d\mathbf{F}(\mathbf{0})^{-1}[\mathbf{F}(\mathbf{p}) - \mathbf{q}] \tag{7.16}$$

[For simplicity we have replaced $d\mathbf{F}(\mathbf{p}_1)$ in (7.15) by $d\mathbf{F}(\mathbf{0})$.] It is shown below, in Lemma 3 that for $\mathbf{q}$ sufficiently small, T_q is a contraction in a neighborhood of $\mathbf{0}$. Thus, for each $\mathbf{q}$ near $\mathbf{0}$, T_q has a unique fixed point, which we denote $\mathbf{G}(\mathbf{q})$. Clearly, $\mathbf{F}(\mathbf{p}) = \mathbf{q}$ if and only if $\mathbf{p}$ is the fixed point of T_q, that is, if and only if $\mathbf{p} = \mathbf{G}(\mathbf{q})$. It remains only to verify that $\mathbf{G}$ is differentiable.

Let $\mathbf{q}_0$ be a point near $\mathbf{0}$, and $\mathbf{p}_0 = \mathbf{G}(\mathbf{q}_0)$. Let $T = d\mathbf{F}(\mathbf{p}_0)$. Then, by definition

$$\mathbf{F}(\mathbf{p}) - \mathbf{F}(\mathbf{p}_0) = T(\mathbf{p} - \mathbf{p}_0) + \boldsymbol{\phi}(\mathbf{p} - \mathbf{p}_0) \tag{7.17}$$

where $\|\boldsymbol{\phi}(\mathbf{p} - \mathbf{p}_0)\| \leq \varepsilon(\mathbf{p} - \mathbf{p}_0)\,\|\mathbf{p} - \mathbf{p}_0\|$ and $\varepsilon(t) \to 0$ as $t \to 0$. Let $\mathbf{p} = \mathbf{G}(\mathbf{q})$. Then (7.17) becomes

$$\mathbf{q} - \mathbf{q}_0 = T(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) + (\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0))$$

Since T is invertible this can be rewritten as

$$\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0) = T^{-1}(\mathbf{q} - \mathbf{q}_0) + T^{-1}\boldsymbol{\phi}(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) \tag{7.18}$$

If we can successfully study the behavior of the last term we will have verified the differentiability of $\mathbf{G}$ at $\mathbf{q}_0$, with

$$d\mathbf{G}(\mathbf{q}_0) = T^{-1} = d\mathbf{F}(\mathbf{p}_0)^{-1}$$

But (7.18) gives us

$$\|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)\| \le \|T^{-1}\| \, \|\mathbf{q} - \mathbf{q}_0\| + \|T^{-1}\| \varepsilon(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) \, \|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)\| \tag{7.19}$$

Since $\mathbf{G}$ is continuous (by Problem 10), we may choose $\mathbf{q}$ so close to $\mathbf{q}_0$ that the last term is dominated by $1/2 \, \|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)\|$. Then (7.19) is the same as

$$\|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)\| \le 2 \, \|T^{-1}\| \, \|\mathbf{q} - \mathbf{q}_0\|$$

and (7.18) produces this inequality which guarantees differentiability

$$\begin{aligned} \|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0) - T^{-1}(\mathbf{q} - \mathbf{q}_0)\| &\le \|\boldsymbol{\phi}(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0))\| \\ &\le \varepsilon(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) \, \|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)\| \\ &\le 2 \, \|T^{-1}\| \, \varepsilon(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) \, \|\mathbf{q} - \mathbf{q}_0\| \end{aligned}$$

and certainly $\lim_{\mathbf{q} \to \mathbf{q}_0} 2 \, \|T^{-1}\| \, \varepsilon(\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}_0)) = 0$.

Here is the lemma which guarantees that the T_q are contractions for $\mathbf{q}$ near enough to $\mathbf{0}$:

Lemma 1. *Given the hypotheses of Theorem 7.2, there is a $\delta > 0$ such that for $\mathbf{q} \in B(\mathbf{0}, \delta)$, the map*

$$T(\mathbf{p}) = \mathbf{p} - d\mathbf{F}(\mathbf{0})^{-1}(\mathbf{F}(\mathbf{p}) - \mathbf{q})$$

is a contraction on $B(\mathbf{0}, \delta)$.

Proof. Let $\mathbf{p}, \mathbf{p}'$ be two points near $\mathbf{0}$ and consider the function

$$\mathbf{h}(t) = \mathbf{p} + t(\mathbf{p}' - \mathbf{p}) - d\mathbf{F}(\mathbf{0})^{-1}(\mathbf{F}(\mathbf{p} + t(\mathbf{p}' - \mathbf{p})) - \mathbf{q}) \qquad 0 \le t \le 1$$

Then

$$T(\mathbf{p}) - T(\mathbf{p}') = \mathbf{h}(1) - \mathbf{h}(0) = \int_0^1 \mathbf{h}'(t) \, dt \tag{7.20}$$

$$\mathbf{h}'(t) = \mathbf{p}' - \mathbf{p} - d\mathbf{F}(\mathbf{0})^{-1} \, d\mathbf{F}(\mathbf{p} + t(\mathbf{p}' - \mathbf{p}))(\mathbf{p}' - \mathbf{p})$$

$$\mathbf{h}'(t) = [\mathbf{I} - d\mathbf{F}(\mathbf{0})^{-1} \, d\mathbf{F}(\mathbf{p} + t(\mathbf{p}' - \mathbf{p}))](\mathbf{p}' - \mathbf{p}) \tag{7.21}$$

Now choose $\delta < \mathbf{0}$ so that

$$\|\mathbf{I} - d\mathbf{F}(\mathbf{0})^{-1} \, d\mathbf{F}(\mathbf{x})\| < 1/2$$

for $\|\mathbf{x}\| < \delta$. Then if $\mathbf{p}, \mathbf{p}' \in B(\mathbf{0}, \delta)$, every $\mathbf{p} + t(\mathbf{p}' - \mathbf{p})$ is in $B(\mathbf{0}, \delta)$, for $0 \le t \le 1$, so, using (7.10)

$$\begin{aligned} \|\mathbf{h}'(t)\| &\le \|\mathbf{I} - d\mathbf{F}(\mathbf{0})^{-1}\, d\mathbf{F}(\mathbf{p} + t(\mathbf{p}' - \mathbf{p}))\|\ \|\mathbf{p}' - \mathbf{p}\| \\ &\le \tfrac{1}{2} \|\mathbf{p}' - \mathbf{p}\| \end{aligned}$$

Thus, by (7.9)

$$\|T(\mathbf{p}) - T(\mathbf{p}')\| \le \frac{1}{2} \int_0^1 \|\mathbf{p}' - \mathbf{p}\|\, dt \le \frac{1}{2} \|\mathbf{p}' - \mathbf{p}\|$$

so T is a contraction in $B(\mathbf{0}, \delta)$.

• EXERCISES

4. Compute the Jacobian

$$\frac{\partial(u^1, \ldots, u^n)}{\partial(x^1, \ldots, x^n)}$$

for each of the following functions and determine those points $(x^1, \ldots, x^n)$ at which $u^1, \ldots, u^n$ are coordinates:

(a) $u = xe^y$, $v = ye^x$

(b) $u^1 = x^1 + x^2 + x^3$, $u^2 = x^1x^2 + x^2x^3 + x^3x^1$, $u^3 = x^1x^2x^3$

(c) $u = x^2 - y^2$, $v = xy$

(d) $u = x^2 + y^2 + z^2$, $v = yx^{-1}$, $w = zx^{-1}$

(e) $u^1 = x^1$, $u^2 = \dfrac{x^2}{x^1}$, $\ldots$, $u^n = \dfrac{x^n}{x^1}$

(f) $u^1 = h^1(x)x^n$, $\ldots$, $u^n = h_n(x)x^n$, $h_i(x) \ne 0$ for all i and x

5. Express the differential of $f(\mathbf{x}) = \sum_{i=1}^n (x^i)^2$ in terms of the coordinates $u^1, \ldots, u^n$ given in Exercise 4(e).

6. Express df in terms of the coordinates of Exercise 4(d), where
   (a) $f(x) = \ln(x^2 + y^2 + z^2)$
   (b) $f(x) = yz$
   (c) $f(x) = x + y + z$

7. Compute the differential of

$$[(x-a)^2+(y-b)^2+(z-c)^2]^{-1}$$

in spherical coordinates in $R^3-\{(a, b, c)\}$.

8. What is the rate of change of the volume of a rectangular box with respect to the area of its surface, assuming the length of one side and the sum of the lengths of the other two sides is left fixed?

• PROBLEMS

5. Let f be a differentiable function defined on a domain D in R^2. Show that f is a function of $x+y$ alone if and only if $\partial f/\partial x = \partial f/\partial y$ on D. (*Hint*: Consider the change of coordinates $u = x+y$, $v = x-y$.)

6. Give a condition guaranteeing that a differentiable function of two variables can be expressed as a function of xy.

7. Suppose that f, g are two differentiable functions on R^2 with $\nabla g \neq 0$. Show that f is a function of g alone if and only if ∇f, ∇g are everywhere collinear.

8. Show that for any twice differentiable function f defined on the plane,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{\partial^2 f}{\partial \theta^2}$$

9. Show that for $f(z)=z^n$, $n\neq 0$,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

10. The proof of Theorem 7.2 is still incomplete: we must show that the function $\mathbf{G}$ is continuous. There are two ways.

(a) Suppose $\mathbf{q}_n \to \mathbf{q}$. Let $\mathbf{q}_n = \mathbf{F}(\mathbf{p}_n)$. Suppose that $\mathbf{p}_n \to \mathbf{p}$. Then $\mathbf{F}(\mathbf{p}) = \lim \mathbf{F}(\mathbf{p}_n) = \lim \mathbf{q}_n = \mathbf{q}$. Applying $\mathbf{G}$ we have

$$\lim \mathbf{G}(\mathbf{q}_n) = \lim \mathbf{p}_n = \mathbf{p} = \mathbf{GF}(\mathbf{p}) = \mathbf{G}(\mathbf{q})$$

Thus $\mathbf{G}$ is continuous, as desired. Why may we suppose that the sequence $\{\mathbf{p}_n\}$ converges?

(b) In this approach we reprove Theorem 7.2 so that the continuity is automatic.

For a sufficiently small $\varepsilon > 0$, we consider the space C of continuous functions h on $\{\mathbf{q} \in R^n\colon \|\mathbf{q}\| \leq \varepsilon\}$ such that $h(\mathbf{0}) = \mathbf{0}$. Define $T\colon C \to C$ by

$$T(\mathbf{h})(\mathbf{q}) = \mathbf{h}(\mathbf{q}) - d\mathbf{F}(\mathbf{0})^{-1}[\mathbf{F}(\mathbf{h}(\mathbf{q})) - \mathbf{q}]$$

As in Lemma 3 show that T is a contraction (on the space C of continuous functions!). Thus T has a fixed point $\mathbf{G}$. Clearly, $\mathbf{F}(\mathbf{G}(\mathbf{q})) = \mathbf{q}$ as desired and the continuity of $\mathbf{G}$ is assured.

11. Suppose that f is a continuously differentiable function defined in a neighborhood of 0 in R^3, and $f(0) = 0$ and $\partial f/\partial z(0) \neq 0$. Then the equation

$$f(x, y, z) = 0$$

implicitly defines z as a function of x and y. More precisely, there is a function g defined for small enough x, y such that

$$f(x, y, z) = 0 \quad \text{if and only if} \quad z = g(x, y)$$

near the origin. This can be proven as a corollary of Theorem 7.2 as follows: applying Theorem 7.2 to the mapping

$$\begin{aligned} u &= x \\ v &= y \\ w &= f(x, y, z) \end{aligned} \tag{7.22}$$

We can find functions h, k, g of (u, v, w) such that (7.22) holds if and only if

$$\begin{aligned} x &= h(u, v, w) \\ y &= k(u, v, w) \\ z &= g(u, v, w) \end{aligned}$$

Obviously, $h(u, v, w) = u$, $k(u, v, w) = v$. It follows that when $w = 0$, $z = g(u, v, 0) = g(x, y, 0)$. This is the desired conclusion.

12. Here is a similar fact. The proof should be analogous to the argument for Problem 11. Suppose f, g are continuously differentiable near $\mathbf{0}$ in R^3 and that $f(\mathbf{0}) = g(\mathbf{0}) = 0$ and

$$\det \begin{pmatrix} \dfrac{\partial f}{\partial x}(0) & \dfrac{\partial f}{\partial y}(0) \\ \dfrac{\partial g}{\partial x}(0) & \dfrac{\partial g}{\partial y}(0) \end{pmatrix} \neq 0$$

Then, there are continuously differentiable functions h, k defined for small enough z such that

$$f(x, y, z) = 0 = g(x, y, z) \quad \text{if and only if} \quad x = h(z) \quad y = k(z)$$

7.3 Differential Forms

The differential of a real-valued function defined on a domain D in R^n is a function defined on D whose values are linear functions on R^n. A function of this type is called a *differential form*, and a central issue in the calculus of several variables is this: just when is a differentiable form the differential of a function? This problem is resolved by the generalization of the fundamental theorem of calculus which takes the form in this chapter of Green's theorem. The one-variable fundamental theorem asserts that every differential form on an interval is the differential of a function. This is far from being true in several variables.

For example, to say that $\sum a_i\, dx^i$ is the differential of a function f is to assert that $a_i = \partial f/\partial x^i$. Since

$$\frac{\partial^2 f}{\partial x^i\, \partial x^j} = \frac{\partial^2 f}{\partial x^j\, \partial x^i} \qquad \text{all } i \text{ and } j \tag{7.23}$$

we must have $\partial a_j/\partial x^i = \partial a_i/\partial x^j$. This is not always the case. $e^x(dx + dy)$, $ydx - xdy$ are *not* differentials of functions because the coefficients do not satisfy these conditions. We shall explore this situation at length in the following two sections.

Definition 3. Let D be a domain in R^n. A **differential form** on D is a function which associates to each point $\mathbf{p}$ in D a linear functional on R^n.

If f is a differentiable function on D, the df is a differential form on D. In particular, if $x_1, \ldots, x_n$ is a coordinate system in R^n, $dx_1, \ldots, dx_n$ are differential forms on R^n. Furthermore, for any $\mathbf{p} \in D$, $dx_1(\mathbf{p}), \ldots, dx_n(\mathbf{p})$ form a basis of the space of linear functionals on R^n, so any such functional is a linear combination of the $dx_i(\mathbf{p})$. Thus, the general differential form on D is of the form $\sum_{i=1}^n a_i(\mathbf{p})\, dx_i(\mathbf{p})$ where the a_i are real-valued functions on D.

Definition 4. Let ω be a differential form on the domain D, and write $\omega = \sum a_i\, dx_i$ relative to the standard coordinates of R^n. We shall say that ω is a k-**times (continuously) differentiable differential form on** D ($\omega \in C^k(D)$) if the functions $a_1, \ldots, a_n$ are all k-times (continuously) differentiable.

Suppose now that $u_1, \ldots, u_n$ are differentiable functions in $D \subset R^n$ and that $du_1(\mathbf{p}), \ldots, du_n(\mathbf{p})$ are independent at some $\mathbf{p} \in R^n$. Such an n-tuple of function forms a coordinate system near $\mathbf{p}$: the mapping $\mathbf{u} = (u_1, \ldots, u_n)$

maps a neighborhood D of p onto a domain D' in one-to-one fashion. Furthermore, $du_1(\mathbf{p}), \ldots, du_n(\mathbf{p})$ forms a basis for $(R^n)^*$, so any differential form can be written as $\sum \alpha_i \, du_i$. We can compute the relation between the α_i and the a_j by the chain rule: since $du^i = \sum \partial u^i/\partial x^j \, dx^j$, we have

$$a_j = \sum_{i=1}^{n} \alpha_i \frac{\partial u^i}{\partial x^j} \tag{7.24}$$

Thus differential forms transform under a coordinate change as the differential of a function (compare Equations (7.4) and (7.23)). Now the equality of mixed partials of a twice differentiable function gives a necessary condition for a differential form to be the differential of a function.

Proposition 3. *Let ω be a continuously differentiable differential form in a domain D. Suppose $u^1, \ldots, u^n$ is any coordinate system for D. If $\omega = \sum a_i \, du^i$ is the differential of a function we must have*

$$\frac{\partial a_i}{\partial u^j} = \frac{\partial a_j}{\partial u^i} \qquad 1 \le i, j \le n \tag{7.25}$$

Proof. If $\omega = df$, then $a_i = \partial f/\partial u^i$. Then

$$\frac{\partial a_i}{\partial u^j} = \frac{\partial}{\partial u^j}\left(\frac{\partial f}{\partial u^i}\right) = \frac{\partial}{\partial u^i}\left(\frac{\partial f}{\partial u^j}\right) = \frac{\partial u^i}{\partial a_j}$$

Closed and Exact Forms

We shall say that a differential form is **exact** in a domain D if it is the differential of a function, and **closed** if Equations (7.25) hold. It is easily verified that if these equations hold in any coordinate system, then they hold in all coordinate systems (see Problem 13); so it is not too difficult to verify that a form is closed.

In the plane a form has the expression $\omega = p \, dx + q \, dy$ with respect to the rectangle coordinates. In this case there is only one nontrivial equation in (7.25), namely,

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0 \tag{7.26}$$

We shall refer to this function as $d\omega$; that is, if

$$\omega = p \, dx + q \, dy \qquad d\omega = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

Thus, a differential form on a plane is closed if $d\omega = 0$ and is exact if $\omega = df$.

Examples

14. $\omega = x\,dx + y\,dy$, $d\omega = 1 - 1 = 0$. In fact, $\omega = d(x^2 + y^2)/2$.

15. $\omega = y\,dx + x\,dy$, $d\omega = 1 - 1 = 0$. Here ω is also exact, since $\omega = d(xy)$.

16. $\omega = y\,dx - x\,dy$, $d\omega = -1 - 1 = -2$, so ω is not closed. Notice however that $y^{-2}\omega$ is closed, since it is exact (except for $y = 0$, where it is not defined), $y^{-2}\omega = d(x/y)$.

17. **Integrating factors.** Let $\omega = p\,dx + g\,dy$ be a differential form given in a neighborhood of $\mathbf{p}_0$. The vector field $(-q, p)$ can be realized as the field of tangents to a family of curves, as we saw in Chapter 4. Let this family be given implicitly by

$$F(x, y) = c$$

Thus, since $F(x, y)$ is constant on these curves, its derivative along the curve is zero; or what is the same

$$dF(x, y)(-q(x, y), p(x, y)) = 0$$

Since dF and ω annihilate the same vectors at each point, they are collinear. Thus there is a function $\lambda(x, y)$ such that

$$dF = \lambda\omega$$

We conclude that for any differential form ω there is a factor λ such that $\lambda\omega$ is exact. This is true in two dimensions, and fails in higher dimensions. λ is called an **integrating factor** for ω.

18. The polar coordinate θ is *not* a well-defined function on the domain $R^2 - \{\mathbf{0}\}$, but its differential is:

$$d\theta = d\left(\arctan\frac{y}{x}\right) = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

Thus this form is closed, but not exact on the domain $R^2 - \{\mathbf{0}\}$.

We shall now verify that every closed form on $R^2 - \{\mathbf{0}\}$ is equal to an exact form plus a constant multiple of $d\theta$. Thus the space of closed forms on $R^2 - \{\mathbf{0}\}$ is larger than the space of exact forms by one dimension.

Suppose that ω is a closed form in $R^2 - \{\mathbf{0}\}$. In polar coordinates $\omega(re^{i\theta}) = a(re^{i\theta})dr + b(re^{i\theta})d\theta$ and since ω is closed we have $\partial b/\partial r = \partial a/\partial\theta$. It follows that

$$F(r) = \int_0^{2\pi} b(re^{i\theta})\, d\theta$$

is a constant. For

$$\frac{dF}{dr} = \int_0^{2\pi} \frac{\partial b}{\partial r}\, d\theta = \int_0^{2\pi} \frac{\partial a}{\partial \theta}\, d\theta = a(re^{2\pi i}) - a(re^0) = a(r) - a(r) = 0$$

Let $c(\omega)$ be that constant. Notice that $c(d\theta) = 2\pi$. Further, if $\omega = df$, then $c(\omega) = 0$. For

$$c(\omega) = \int_0^{2\pi} b\, d\theta = \int_0^{2\pi} \frac{\partial f}{\partial \theta}\, d\theta = f(re^{2\pi i}) - f(re^0) = 0$$

Conversely, if $c(\omega) = 0$, then ω is the differential of a function defined on $R^2 - \{\mathbf{0}\}$. Let

$$f(r, \theta) = \int_1^r a(t)\, dt + \int_0^\theta b(re^{i\phi})\, d\phi \tag{7.27}$$

Since $c(\omega) = 0$, $f(r, \theta + 2\pi) = f(r, \theta)$ for all r, θ, so we can define a function F on $R^2 - \{\mathbf{0}\}$ by $F(re^{i\theta}) = f(r, \theta)$. Differentiating (7.26), we have

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial r} = a(r) + \int_0^\theta \frac{\partial b}{\partial r}(re^{i\phi})\, d\phi = a(r) + \int_0^\theta \frac{\partial a}{\partial \phi}(re^{i\phi})\, d\phi = a(re^{i\theta})$$

$$\frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial \theta} = b(re^{i\theta})$$

Thus, $dF = \omega$.

Finally, if ω is any closed form on $R^2 - \{\mathbf{0}\}$, let $\theta = \omega - c(\omega)d\theta/2\pi$. Then

$$c(\theta) = c(\omega) - \frac{c(\omega)}{2\pi} 2\pi = 0$$

so θ is exact: $\theta = dF$. Thus

$$\omega = dF + \frac{c(\omega)}{2\pi}\, d\theta$$

• EXERCISES

9. Which of the following forms are closed?

(a) $\sum_{i=1}^{n} x^i\, dx^i$
(b) $xy\, dz + yz\, dx + zx\, dy$
(c) $xyz(dx + dy + dz)$
(d) $r\, dr + d\theta$
(e) $r^2\, dr + r\, d\theta$
(f) $r \sin\theta\, dr + r\cos\theta$
(g) $r\sin\phi\, dr + r\cos\phi\sin\theta\, d\theta + r\sin\phi\, d\phi$
(h) $d(xe^{yz}\cos(xyz))$
(i) $x_1x_2\, dx_3 + x_2x_3\, dx_4 + x_3x_4\, dx_1 + x_4x_1\, dx_2$
(j) $x_1\, dx_2 + x_3\, dx_4 + x_5\, dx_6$
(k) $x_1\, dx_2 + x_2\, dx_3 + x_3\, dx_1$

10. Is the form $(z - a)^{-1}\, dz$ exact in $C - \{a\}$? Is its real part exact? Is its imaginary part exact?

11. Find integrating factors for the following forms:

(a) $x(dy + dx)$
(b) $xy(dx + dy)$
(c) $-y\, dx + x\, dy$
(d) $x\, dy$
(e) $e^{x+y}\, dx + e^x\, dy$
(f) $\sin x\, dx + \cos x\, dy$

• PROBLEMS

13. Let $(x^1, \ldots, x^n)$ and $(u^1, \ldots, u^n)$ be two coordinate systems valid in a domain D in R^n. Let ω be a differential form defined in D and write ω in terms of these coordinates as

$$\omega = \sum_{i=1}^{n} a_i\, dx^i = \sum_{i=1}^{n} \alpha_i\, du^i$$

Show that if

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i} \qquad \text{for all } i, j$$

then

$$\frac{\partial \alpha_i}{\partial u^j} = \frac{\partial \alpha_j}{\partial u^i} \qquad \text{for all } i, j$$

14. Let the hypotheses be the same as in Problem 13, but this time suppose $n = 2$. Show that

$$\frac{\partial a_1}{\partial x_2} - \frac{\partial a_2}{\partial x_1} = \left(\frac{\partial \alpha_1}{\partial u_2} - \frac{\partial \alpha_2}{\partial u_1}\right)\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}$$

15. Show that the space of closed forms on $R^2 - \{0, 1\}$ is larger than the space of exact forms by 2 dimensions. (*Hint*: Let θ_0 be the ordinary polar angle, and let $\theta_1(p)$ be the angle between the ray from 1 to p and the horizontal. Then if ω is closed in $R^2 - \{0, 1\}$ there are constants a, b such that $\omega - a\, d\theta_0 - b\, d\theta_1$ is exact.)

7.4 Work and Conservative Fields

Suppose we have a field of forces $\mathbf{F}$ given in a domain D in R^n: $\mathbf{F}(\mathbf{x})$ is the force felt by a unit mass situated at the point $\mathbf{x}$. In moving an object of mass m along a certain path a certain amount of energy is expended; this is called *work*. In this section we shall describe the computation of work.

Suppose first that a body of mass m moving in a straight line experiences a force of magnitude F per unit mass operating in the direction opposite the motion. Then, by definition the work required to move that body a distance d is $-F \cdot m \cdot d$. In a more complicated situation the force acts in space in a fixed direction with a certain magnitude; thus the force is represented by a vector $\mathbf{F}$. Suppose we want to move a body of mass m from a point $\mathbf{a}$ to another point $\mathbf{b}$. The work required for this movement will depend only on the component of the force in the direction of motion and will be given again by $-F_0 \cdot m \cdot d$, where F_0 is this component and d is the distance between $\mathbf{a}$ and $\mathbf{b}$. That is, if $\mathbf{b} - \mathbf{a} = d\mathbf{E}$, where $\mathbf{E}$ is a unit vector, then $F_0 = \langle \mathbf{F}, \mathbf{E} \rangle$ and the work is

$$-\langle \mathbf{F}, \mathbf{E} \rangle md = -m \langle \mathbf{F}, \mathbf{b} - \mathbf{a} \rangle$$

Now, in general, the force is not necessarily constant, but varies with position. The general situation is that of a force given by a vector field (vector-valued function) $\mathbf{F}$ on R^3. Suppose that for some perverse reason we desire to move a given body from $\mathbf{a}$ to $\mathbf{b}$ along a particular path Γ. As is customary we try to adapt the above formula to this revised situation by assuming that the force field varies little over small intervals (that is, **F** is continuous) and that the path is very close to being a sequence of straight line segments. Then, we get a reasonable approximation to the total work involved by adding up the work required over each line segment assuming the force is constant there. More precisely, then, we select a very large number of points $\mathbf{a} = \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_s = \mathbf{b}$ numbered sequentially along the path (see Figure 7.2). The work we seek is then approximated by

$$-m\sum_{i=1}^{s}\langle \mathbf{F}(\mathbf{p}_i), \mathbf{p}_i - \mathbf{p}_{i-1}\rangle \tag{7.28}$$

We define the **work** as the limit of all such sums as the maximum of the distance between successive points tends to zero, and we expect that, as usual, the calculus will make that computable.

And it does. Suppose given, for example, a field of force **F** given in a domain D in R^3; then $\mathbf{F} = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x}))$ is an R^3-valued function defined on D. Suppose Γ is an oriented curve in D, given by the parametrization $\mathbf{x} = \mathbf{g}(t) = (g_1(t), g_2(t), g_3(t))$, $a \le t \le b$. We shall now compute the work done in moving a particle of mass m from $\mathbf{g}(a)$ to $\mathbf{g}(b)$. Let $\mathbf{g}(a) = \mathbf{p}_0, \ldots, \mathbf{p}_s = \mathbf{g}(b)$ be a very large number of points situated along Γ. Referring to the parametrization we can write $\mathbf{p}_0 = \mathbf{g}(t_0)$, $\mathbf{p}_1 = \mathbf{g}(t_1), \ldots, \mathbf{p}_s = \mathbf{g}(t_s)$, with $a = t_0 < t_1 < \cdots < t_s = b$. Then the approximate work done is given by

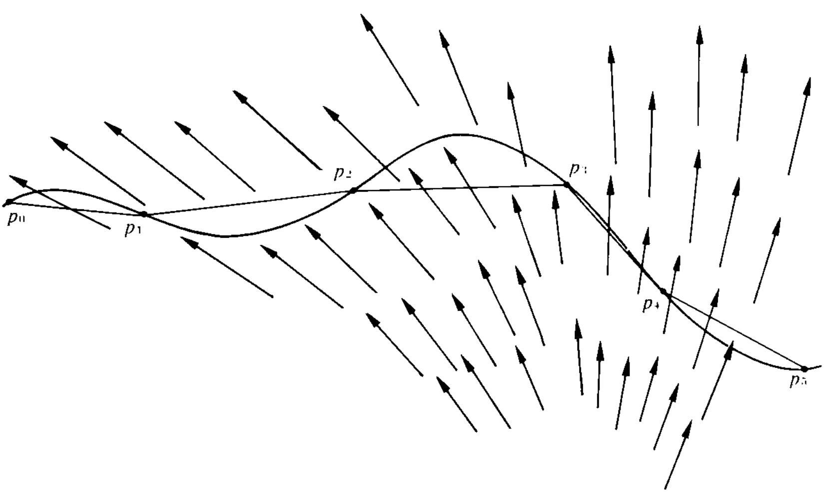


Figure 7.2

(7.28):

$$-m\sum_{i=1}^{s}\langle \mathbf{F}(\mathbf{g}(t_i)), \mathbf{g}(t_i)-\mathbf{g}(t_{i-1})\rangle \tag{7.29}$$

$$= -m\sum_{i=1}^{s} f_i(\mathbf{g}(t_i))[g_1(t_i)-g_1(t_{i-1})] + f_2(\mathbf{g}(t_i))[g_2(t_i)-g_2(t_{i-1})] + f_3(\mathbf{g}(t_i))[g_3(t_i)-g_3(t_{i-1})]$$

By the mean value theorem, there are $\theta_{1,i}, \theta_{2,i}, \theta_{3,i}$ such that

$$g_j(t_i)-g_j(t_{i-1}) = g_j'(\theta_{j,i})(t_i-t_{i-1}) \qquad t_{i-1}\le\theta_{j,i}\le t_i$$

Thus the approximating sum (7.29) becomes

$$-m\sum_{i=1}^{s}\left[\sum_{j=1}^{3} f_j(\mathbf{g}(t_i))g_j'(\theta_{j,i})\right](t_i-t_{i-1}) \tag{7.30}$$

which is a typical Riemann sum approximating

$$-m\int_a^b \sum_{j=1}^{3} f_j(\mathbf{g}(t))g_j'(t)\,dt$$

$$= -m\int_a^b \langle \mathbf{F}(t), \mathbf{g}'(t)\rangle\,dt \tag{7.31}$$

In fact, as the "very large number of points" on Γ becomes infinite, the sums (7.30) do tend to the integral (7.31), so we are justified in referring to this as the work required to move the mass along Γ. We are thus led to this definition of work:

Definition 5. Let D be a domain in R^n and $\mathbf{F}$ a force field defined in D; that is, $\mathbf{F}$ is an R^n-valued function on D. Let Γ be an oriented curve defined in D. The **work required to move a unit mass along Γ** is

$$W(\Gamma, \mathbf{F}) = -\int_a^b \langle \mathbf{F}(t), \mathbf{g}'(t)\rangle\,dt$$

where $\mathbf{g}\colon [a, b] \to \Gamma$ is a parametrization of Γ.

Notice that since $W(\Gamma, \mathbf{F})$ is the limit of a collection of sums defined independently of any particular parametrization that $W(\Gamma, \mathbf{F})$ is also independent of the parametrization.

Sometimes paths of motion have a break in direction (see Figure 7.3). Such a curve is called a **piecewise continuously differentiable curve**, or **a path** for short. More precisely, we make the following definition.

Definition 6. An **oriented path** is the image of an interval $[a, b]$ under a continuous function $\mathbf{f}$ such that

(i) $\mathbf{f}$ is continuously differentiable with nonzero derivative at all but finitely many points $t_1, \ldots, t_s$.
(ii) $\lim\limits_{t \to t_i} \mathbf{f}'(t)$ and $\lim\limits_{t < t_i} \mathbf{f}'(t)$ exist (but are not necessarily equal) and are nonzero.

If $\mathbf{f}(a) = \mathbf{f}(b)$ the path is said to be **closed**. If Γ is an oriented path we can write $\Gamma = \Gamma_1 + \cdots + \Gamma_{s+1}$, where the Γ_i are the oriented curves between the points t_{i-1} and t_i. We define the **work** $W(\Gamma, \mathbf{F})$ by

$$W(\Gamma, \mathbf{F}) = \Sigma W(\Gamma_i, \mathbf{F})$$

Examples

19. Let $\mathbf{F}(x, y) = (-y, x^2)$ be a force field in R^2. The work done by moving a unit mass around the unit circle is found this way. First, we parametrize the circle:

$$\Gamma\colon \mathbf{x} = (\cos t, \sin t) \qquad 0 \le t \le 2\pi$$

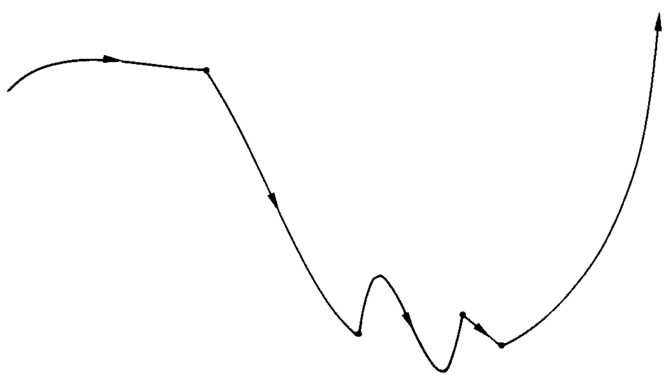


Figure 7.3

Then

$$W(\Gamma, \mathbf{F}) = -\int_0^{2\pi} \langle(-\sin t, \cos^2 t), (-\sin t, \cos t)\rangle\, dt$$

$$= -\int_0^{2\pi} (-\sin^2 t + \cos^3 t)\, dt = \pi$$

20. For the same force field, find the work done around the boundary of Γ of the rectangle $[(0, 0), (1, 1)]$, traversed counterclockwise. Here $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, where

$$\begin{array}{ll} \Gamma_1 : \mathbf{x} = (t, 0) & 0 \le t \le 1 \\ \Gamma_2 : \mathbf{x} = (1, t) & 0 \le t \le 1 \\ \Gamma_3 : \mathbf{x} = (1 - t, 1) & 0 \le t \le 1 \\ \Gamma_4 : \mathbf{x} = (0, 1 - t) & 0 \le t \le 1 \end{array}$$

Then

$$\begin{aligned} W(\Gamma, \mathbf{F}) &= \left\{\int_0^1 \langle(0, t^2), (1, 0)\rangle\, dt + \int_0^1 \langle(t, 1), (0, 1)\rangle\, dt \right. \\ &\quad + \int_0^1 \langle(-1, (1 - t)^2), (-1, 0)\rangle\, dt \\ &\quad \left. + \int_0^1 \langle((1 - t)^2, 0), (0, -1)\rangle\, dt\right\} \\ &= -\int_0^1 (0 + 1 + 1 + 0)\, dt = 2 \end{aligned}$$

21. Let $\mathbf{F}(x, y, z) = (yz, xz, xy)$ and compute the work done along one full loop of the helix

$$\Gamma : \mathbf{x} = (\cos t, \sin t, t) \qquad 0 \le t \le 2\pi$$

$$W(\Gamma, \mathbf{F}) = -\int_0^{2\pi} \langle(t \sin t, t \cos t, \sin t \cos t), (-\sin t, \cos t, 1)\rangle\, dt$$

$$= -\int_0^{2\pi} (-t \sin^2 t + t \cos^2 t + \sin t \cos t)\, dt = 0$$

22. Compute the work done in the presence of the same force field along the curve $x = 1$, $y = 0$, $0 \le z \le 2\pi$. Here Γ is given para-

metrically by

$$\Gamma\colon \mathbf{x} = (1, 0, t) \qquad 0 \le t \le 2\pi$$

Thus

$$W(\Gamma, \mathbf{F}) = -\int_0^{2\pi} \langle (0, t, 0), (0, 0, 1) \rangle \, dt = 0$$

Conservation of Energy

Now let us suppose we are given a field of forces on a domain D. Let Γ be a closed path in D. Under optimal conditions we would hope for no loss of energy in moving a mass around Γ. We shall call a field *conservative* if this situation is the case; that is, the field $\mathbf{F}$ is **conservative** if $W(\Gamma, \mathbf{F}) = 0$ for every closed path Γ. Not every field is conservative, as Examples 19 and 20 show. In case $\mathbf{F}$ is a conservative field, then the work required to move a unit mass from one point $\mathbf{p}_0$ to another $\mathbf{p}_1$ will be the same no matter what path from $\mathbf{p}_0$ to $\mathbf{p}_1$ is followed. For suppose we take two such oriented paths Γ, Γ'. Then the path from $\mathbf{p}_0$ to $\mathbf{p}_0$ obtained by first traversing Γ and then $-\Gamma'$ (Γ' oriented from $\mathbf{p}_1$ to $\mathbf{p}_0$) is a closed path. Thus $W(\Gamma - \Gamma', \mathbf{F}) = 0$ since $\mathbf{F}$ is conservative. But $W(\Gamma - \Gamma', \mathbf{F}) = W(\Gamma', \mathbf{F})$, so

$$W(\Gamma, \mathbf{F}) = W(\Gamma', \mathbf{F})$$

Definition 7. Let $\mathbf{F}$ be a conservative field defined in the domain D. A **potential function** for $\mathbf{F}$ is a real-valued function Π defined on D such that, for any path Γ from $\mathbf{p}$ to $\mathbf{p}'$ we have

$$W(\Gamma, \mathbf{F}) + \Pi(\mathbf{p}') - \Pi(\mathbf{p}) \tag{7.32}$$

is a constant.

Π is sometimes called the **potential energy** of the force field $\mathbf{F}$ and the constancy of (7.32) is just the assertion that a conservative force field obeys the law of conservation of energy. We can relate the potential function of a conservative field with the field, by its differential. We obtain this important result:

Theorem 7.3. *Suppose that D is a domain such that any two points can be joined by a path* (*we say D is pathwise connected*).

(i) *Every conservative field on D has a potential function.*
(ii) *Two potentials of the given field differ by a constant.*

(iii) *If the field* $\mathbf{F} = (f_1, \ldots, f_n)$ *has the potential function* Π, *then*

$$d\Pi = f_1\, dx^1 + \cdots + f_n\, dx^n$$

Proof. (i) Suppose that $\mathbf{F} = (f_1, \ldots, f_n)$ is a conservative field defined on D. Then if Γ and Γ' are two oriented curves with the same end points, $W(\Gamma, \mathbf{F}) = W(\Gamma', \mathbf{F})$ since $\mathbf{F}$ is conservative. Fix $\mathbf{p}_0 \in D$. Since D is arcwise connected, if $\mathbf{p}$ is any point of D there is a curve Γ from $\mathbf{p}_0$ to $\mathbf{p}$. Define $\Pi(\mathbf{p}) = -W(\Gamma, \mathbf{F})$. $\Pi(\mathbf{p})$ is a well-defined function of $\mathbf{p}$ since the work required does not depend on the choice of Γ. Now let $\mathbf{p}$ and $\mathbf{p}'$ be two points in D, and let Γ be a path from $\mathbf{p}$ to $\mathbf{p}'$. If Γ_0 is a curve from $\mathbf{p}_0$ to $\mathbf{p}$, then $\Gamma + \Gamma_0$ is a curve from $\mathbf{p}_0$ to $\mathbf{p}'$, so

$$-W(\Gamma_0, \mathbf{F}) = \Pi(\mathbf{p}), \quad -W(\Gamma + \Gamma_0, \mathbf{F}) = \Pi(\mathbf{p}')$$

But $W(\Gamma + \Gamma_0, \mathbf{F}) = W(\Gamma, \mathbf{F}) + W(\Gamma_0, \mathbf{F}) = W(\Gamma, \mathbf{F}) - \Pi(\mathbf{p})$. Thus $-\Pi(\mathbf{p}') = W(\Gamma, \mathbf{F}) - \Pi(\mathbf{p})$, or $W(\Gamma, \mathbf{F}) + \Pi(\mathbf{p}') - \Pi(\mathbf{p}) = 0$, so (i) is proven.

(ii) If Π' is another potential and Γ_0 is a curve joining $\mathbf{p}_0$ to $\mathbf{p}$ then by the above definition

$$\Pi'(\mathbf{p}) - \Pi'(\mathbf{p}_0) + W(\Gamma_0, \mathbf{F})$$

is a constant, say C. But $W(\Gamma_0, \mathbf{F}) = -\Pi(\mathbf{p})$, by definition, thus

$$\Pi'(\mathbf{p}) - \Pi(\mathbf{p}) = C + \Pi'(\mathbf{p}_0)$$

another constant. Thus two potentials for the field $\mathbf{F}$ indeed differ by a constant.

(iii) Finally, we prove that $d\Pi = \sum f_i\, dx_i$. Let $\mathbf{p} \in D$. Fix i, and let ε be so small that the ball $B(\mathbf{p}, \varepsilon) \subset D$. Let Γ_ε be the curve with this parametrization

$$\mathbf{g}(t) = \mathbf{p} + t\mathbf{E}_i \qquad 0 \le t \le \varepsilon$$

Since Π is a potential for $\mathbf{F}$,

$$\Pi(\mathbf{p} + \varepsilon\mathbf{E}_i) - \Pi(\mathbf{p}) = -\,W(\Gamma_\varepsilon, \mathbf{F}) = \int_0^\varepsilon \langle \mathbf{F}(\mathbf{g}(t)), \mathbf{g}'(t)\rangle\, dt$$

Now $\mathbf{g}'(t) = \mathbf{E}_i$ and

$$\langle \mathbf{F}(\mathbf{g}(t)), \mathbf{g}'(t)\rangle = \sum_j f_j(\mathbf{p} + t\mathbf{E}_i)\, \langle \mathbf{E}_j, \mathbf{E}_i\rangle = f_i(\mathbf{p} + t\mathbf{E}_i)$$

Thus

$$\Pi(\mathbf{p} + \varepsilon\mathbf{E}_i) - \Pi(\mathbf{p}) = \int_0^\varepsilon f_i(\mathbf{p} + t\mathbf{E}_i)\, dt$$

Thus

$$\frac{\partial \Pi}{\partial x_i}(\mathbf{p}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} f_i(\mathbf{p} + t\mathbf{E}_i)\, dt = f_i(\mathbf{p})$$

and so the proof of the theorem is concluded.

• EXERCISES

12. Find the work required to move a unit mass around the given path Γ in the presence of the given force field:

(a) $\mathbf{F}(x, y) = (y, x)$ $\quad\Gamma$: unit circle

(b) $\mathbf{F}(x, y) = (y^2, y - x^2)$ $\quad\Gamma$: boundary of the triangle with vertices at (0, 0), (0, 2), (0, 1)

(c) $\mathbf{F}(x, y) = (1, x)$ $\quad\Gamma: z(t) = \exp(1 + i)t$ from $t = 0$ to $t = 1$

(d) $\mathbf{F}(x, y, z) = (-y, x, z)$ $\quad\Gamma: x = (\cos t, \sin t, t)$

(e) $\mathbf{F}(x, y) = (x, xy)$ $\quad\Gamma$: the portion of the parabola $y = kx^2$ from (0, 0) to (a, ka^2)

(f) $\mathbf{F}(x, y, z) = (z, x^2, y)$ $\quad\Gamma$: closed polygon with successive vertices (0, 0, 0), (2, 0, 0), (2, 3, 0), (0, 0, 1), (0, 0, 0)

13. Which of these fields are conservative?

(a) $\mathbf{F}(x, y) = (\cos x, \cos y, \sin x \sin y)$

(b) $\mathbf{F}(x, y) = (\cos x \cos y, -\sin x \sin y)$

(c) $\mathbf{F}(x, y) = (x, y)$

(d) $\mathbf{F}(x, y, z) = (y, z, x)$

(e) $\mathbf{F}(x, y, z) = (-y, x, 1)$

(f) $-(x^2 + y^2)^{-1/2}(x, y)$

(g) $(x^2 + y^2)^{-1/2}(-y, x)$

• PROBLEMS

16. Let $\mathbf{F}(x, y) = (A(x), B(y))$. Show that $W(\Gamma, \mathbf{F}) = 0$ for any closed path Γ.

17. Find potential functions for these fields:

(a) $\mathbf{F}(x, y, z) = -(0, 0, 1)$

(b) $\mathbf{F}(x, y, z) = -(x^2 + y^2 + z^2)^{-1/2}(x, y, z)$

(c) $\mathbf{F}(x, y, z) = (y, x, 1)$

(d) $\mathbf{F}(x, y, z) = xy\, dz + yz\, dx + zx\, dy$

18. Let $\mathbf{F}$ be a force field in the domain D and Γ an oriented path in D from $\mathbf{p}_0$ to $\mathbf{p}$. Show that the work $W(\Gamma, \mathbf{F})$ can be written as

$$\int_{p_0}^{p} \|\mathbf{F}\| \cos \theta \, ds$$

where s is arc length along Γ_1 and θ is the angle between $\mathbf{F}$ and the tangent to Γ.

19. Suppose the field $\mathbf{F}$ has the potential function Π. The surfaces $\Pi = \text{constant}$ are called **equipotential surfaces** for the field $\mathbf{F}$.

(a) What are the equipotential surfaces for a central force field?

(b) What are the equipotentials for the fields of Exercise 13 which are conservative?

20. Show that if $\mathbf{F}$ is a conservative force field in R^2 the lines of force for $\mathbf{F}$ are orthogonal to the equipotential curves for $\mathbf{F}$.

21. If $\mathbf{F}$ is a vector field in a domain D in the plane, we define $^*\mathbf{F}$ as the field perpendicular and clockwise to $\mathbf{F}$ of the same magnitude. Verify this relation between $\mathbf{F}$ and $^*\mathbf{F}$ if

$$\mathbf{F}(\mathbf{p}) = (A_1(p), A_2(p)), \quad {}^*\mathbf{F} = (-A_2(\mathbf{p}), A_1(\mathbf{p}))$$

22. Suppose both $\mathbf{F}$ and $^*\mathbf{F}$ are conservative fields with potentials Π, Π^*, respectively.

(a) Π is harmonic.

(b) $\Pi + i\Pi^*$ satisfies the Cauchy–Riemann equations.

23. If $f = u + iv$ is a complex analytic function, u is the potential for a field $\mathbf{F}$ such that $^*\mathbf{F}$ is also conservative (and has potential function v).

24. A vector field $\mathbf{F}$ is called **radial** if it is central and its magnitude is a function of the radius. Show that if $\mathbf{F}$ is a nonzero radial vector field it is conservative, but $^*\mathbf{F}$ is *not*.

7.5 Integration of Differential Forms

The study of work has led us to differentials of function via the obvious relation between vector fields and differential forms. If $\mathbf{F} = (f_1, \ldots, f_n)$ is a vector field defined on a domain in R^n, the differential form $\sum_{i=1}^n f_i\, dx^i$ will be denoted $\langle \mathbf{F}, d\mathbf{x}\rangle$ (for obvious reasons). According to the results of Section 7.4, the field $\mathbf{F}$ is conservative if and only if the form $\langle \mathbf{F}, d\mathbf{x}\rangle$ is exact. In this case $\langle \mathbf{F}, d\mathbf{x}\rangle = d(\Pi)$, where Π is a potential function for the field $\mathbf{F}$.

On the other hand, if ω is a form we can write $\omega = \langle \mathbf{F}, d\mathbf{x}\rangle$ for some vector field $\mathbf{F}$ (if $\omega = \sum a_i\, dx^i$, $\mathbf{F} = (a_1, \ldots, a_n)$). We can thus rely on the notion of work to define the integral of ω over a path Γ:

$$\int_\Gamma \omega = \int_\Gamma \langle \mathbf{F}, d\mathbf{x}\rangle = -W(\Gamma, \mathbf{F}) \tag{7.33}$$

Thus, if $\omega = \sum a_i\, dx^i$ and Γ is parametrized explicitly by $x^i = x^i(t)$ for $a \le t \le b$, then

$$\int_\Gamma \omega = \int_a^b \sum_i a_i \frac{dx^i}{dt}\, dt \tag{7.34}$$

The idea of defining the integral of a form in terms of work presents us with a subtle inconsistency which we would like to avoid. The notion of a differential form on R^n involves the geometry of R^n only insofar as it is a vector space. In the conception of differential form, the inner product of R^n is irrelevant and no particular coordinatization of R^n is selected over any other. But the notion of work is deliberately expressed in terms of the Euclidean structure of R^n; it essentially involves lengths and angles. As a result, with the definition (7.33) of integration, we can only compute the integral by means of (7.34) in terms of rectangular coordinates for R^n. Since the concept of differential form is free of a particular basis, we want accessory concepts (such as integration) also to be free; in fact, we would hope to compute $\int_\Gamma \omega$ by means of (7.34) with respect to *any* coordinate system as well as *any* parametrization of Γ. This turns out to be the case, and therein we begin to see the importance of the notion of invariance with respect to coordinate choices.

Proposition 4. *Let ω be a differential form defined on a domain D in R^n and suppose $\omega = \sum f_i \, dx^i = \sum \phi_i \, du^i$ with respect to two different coordinate systems $(x^1, \ldots, x^n)$, $(u^1, \ldots, u^n)$. Let Γ be a path in D parametrized in two different ways by*

$$\begin{aligned} x^i &= x^i(t) \qquad && a \le t \le b \\ u^i &= u^i(\tau) \qquad && \alpha \le \tau \le \beta \end{aligned}$$

Then

$$\int_a^b \sum f_i(\mathbf{x}(t)) \frac{dx^i}{dt} \, dt = \int_\alpha^\beta \sum \phi_i(\mathbf{u}(\tau)) \frac{du^i}{d\tau} \, d\tau \tag{7.35}$$

Proof. We can write the x's as functions of the u's and t as a function of τ.

$$\begin{aligned} & x^i = x^i(u^1, \ldots, u^n) \qquad \text{in } D \\ & t = t(\tau) \qquad \alpha \le \tau \le \beta \end{aligned}$$

Now, according to (7.24)

$$\phi_i(\mathbf{u}) = \sum_{j=1}^n f_j(\mathbf{x}) \frac{\partial x^j}{\partial u^i}(\mathbf{u}) \tag{7.36}$$

when x, u are coordinates for the same point.

Now, let us compute the integral on the left of (7.35) by the change of coordinates $t \to \tau$, according to the calculus of one dimension.

$$\int_a^b \sum_j f_j(\mathbf{x}(t)) \frac{dx^j}{dt} \, dt = \int_\alpha^\beta \sum f_j(\mathbf{x}(t(\tau)) \frac{dx^j}{dt} \frac{dt}{d\tau} \, d\tau = \int_\alpha^\beta \sum_j f_j(\mathbf{x}(t(\tau)) \frac{dx^j}{d\tau} \, d\tau \tag{7.37}$$

But we can compute $dx^j/d\tau$ by the chain rule; $\mathbf{x}$ is a function of $\mathbf{u}$ which is a function of τ:

$$\frac{dx^j}{d\tau} = \sum_i \frac{\partial x^j}{\partial u^i} \frac{du^i}{d\tau}$$

(7.37) becomes

$$\int_\alpha^\beta \sum_{i,j} f_j(\mathbf{x}(t(\tau))) \frac{\partial x^j}{\partial u^i} \frac{du^i}{d\tau} \, d\tau = \int_\alpha^\beta \phi_i(\mathbf{u})(\tau)) \frac{du^i}{d\tau} \, d\tau$$

by (7.36). The proof is concluded.

On the basis of that proposition we may now define the path integral of a form.

Definition 8. (The Path Integral) Let Γ be an oriented path in a domain in which the form ω is defined. If $\Gamma = \sum_{i=1}^s \Gamma_i$, where the Γ_i are parametrized by $\mathbf{x} = \mathbf{g}_i(t)$, $a_i \le t \le b_i$, we define

$$\int_\Gamma \omega = \sum_{i=1}^s \int_{a_i}^{b_i} \omega(\mathbf{g}_i(t), (\mathbf{g}_i'(t)) \, dt$$

Notice, that if Γ is parametrized with respect to arc length, then $\mathbf{g}'$ is the tangent and the integral may be written as

$$\int_\Gamma \omega = \int_\Gamma \omega(\mathbf{T}) \, ds$$

Examples

23. Find $\int_\Gamma r^2 \, d\theta$, where Γ is the boundary of the rectangle $-1 \le x \le 1$, $-1 \le y \le 1$. Now, in rectangular coordinates $r^2 \, d\theta = -y \, dx + x \, dy$. Thus

$$\int_\Gamma r^2 \, d\theta$$

$$= \int_{-1}^1 -(-1) \, dx + \int_{-1}^1 (1) \, dy - \int_{-1}^1 -(1) \, dx - \int_{-1}^1 (-1) \, dy = 8$$

24. Find $\int_\Gamma (x^2 + y^2 + z^2)(dx + xy \, dy + dz)$ around the curve

$x^2 + y^2 = a^2$, $x^2 + y^2 + z^2 = b^2$. This can be parametrized by

$$x = a\cos\theta \qquad y = a\sin\theta \qquad z = (b^2 - a^2)^{1/2}$$

and thus has two branches. Thus

$$\int_\Gamma (x^2 + y^2 + z^2)(dx + xy\,dy + dz)$$
$$= 2\int_0^{2\pi} (-a\sin\theta\,d\theta + a^2\cos^2\theta\sin\theta\,d\theta) = 0$$

In case the curve Γ is a closed path (a continuous image of a circle) it is customary to write $\oint_\Gamma$ to indicate that the integration is around a loop. We now summarize what we know so far about the integration of differential forms.

Theorem 7.4. *Let $\omega = \sum a_i\,dx^i$ be a differentiable differential form defined on a domain D in R^n.*

(i) *ω is the differential of a function if and only if $\oint_\Gamma \omega = 0$ for all closed curves Γ.*

(ii) *ω is the differential of a function if and only if the field $(a_1, \ldots, a_n)$ is conservative.*

(iii) *If $\omega = df$, then*

$$\frac{\partial a_i}{\partial x_j}(\mathbf{p}) = \frac{\partial a_j}{\partial x_i}(\mathbf{p}) \qquad \text{all } i, j \quad \text{all } \mathbf{p} \in D \tag{7.38}$$

When is a Closed Form Exact?

For certain domains, Equations (7.38) are sufficient to guarantee that the form ω is the differential of a function; but this is not always true. For example, let

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2} \qquad \text{in } R^2 - \{(0, 0)\}$$

Certainly, ω satisfies the required conditions (recall Example 5):

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)$$

If ω were the differential of a function, then we would have $\int_\Gamma \omega = 0$ for every closed curve Γ. However since $\omega = d\theta$ (as remarked in Example 5), $\int_\Gamma \omega = 2\pi$ if Γ is a circle centered at the origin. Notice that in some sense ω is the differential of a function, albeit not single valued. If we exclude the line $x = 0$ (or the line $y = 0$), in the remaining domain we can take a principal value of $\theta = \tan^{-1} y/x$; but we cannot find a *continuous single-valued* function on all of $R^2 - \{0, 0\}$ whose differential is ω.

Of course, in the above example in any small enough neighborhood of any point in $R^2 - \{(0, 0)\}$ we can write $\omega = df$ for some function f. This is in fact true for any differential form satisfying the compatability equations (7.38). That is, suppose $\omega = \sum a_i\, dx_i$ is a differentiable differential form defined in a neighborhood U of $\mathbf{p}_0$ in R^n and the equations (7.38) are satisfied. Then if B is a ball centered at $\mathbf{p}_0$ and contained in U, there is a differentiable function f defined in B such that $df = \omega$ in B. This is really easy to prove: if $\mathbf{p}$ is any point in B, let L_p be the oriented line from $\mathbf{p}_0$ to $\mathbf{p}$ and define $f(\mathbf{p}) = \int_{L_p} \omega$. Then, we can differentiate f with respect to x^j by differentiating under the integral sign:

$$\frac{\partial f}{\partial x^j} = \frac{\partial}{\partial x^j} \int_{L_p} \omega = \int \frac{\partial}{\partial x^j} \left(\sum_i a_i\, dx^i \right)$$

Now the integrand will have one term of the form $\sum_i \partial a_i/\partial x^j\, dx^i$, which is by Equations (7.38) the same as $\sum_j \partial a_j/\partial x^i\, dx^i = da_j$. This is the essential term: by the fundamental theorem of calculus we can conclude from $\partial f/\partial x^j = \int da_j$ that $\partial f/\partial x^j = a_j$ as desired. Here is the precise proof.

Theorem 7.5. (Poincaré's Lemma) *Suppose that D is a domain with this property: there is a $\mathbf{p}_0 \in D$ such that for every $\mathbf{p} \in D$ the line joining $\mathbf{p}$ to $\mathbf{p}_0$ is also in D. (D is star shaped (see Figures 7.4 and 7.5).) Then in D every closed form is exact.*

Proof. We may suppose $\mathbf{p}_0$ is the origin. For $\mathbf{p} \in D$, let L_p be the oriented line segment joining $\mathbf{0}$ to $\mathbf{p}$. We may parametrize L_p by

$$L_p: \mathbf{x} = \mathbf{x}(t) = t\mathbf{p} \qquad 0 \le t \le 1 \tag{7.39}$$

If ω is a closed form, define $f(\mathbf{p}) = \int_{L_p} \omega$. We shall show that $df = \omega$. In coordinates, $\mathbf{p} = (x^1, \ldots, x^n)$, $\omega = \sum a_i\, dx^i$, and by (7.39)

$$f(x^1, \ldots, x^n) = \int_{L_p} \sum a_i \frac{dx^i}{dt}\, dt = \int_0^1 \sum_{i=1}^n a_i\, (t\mathbf{x}) x^i\, dt$$

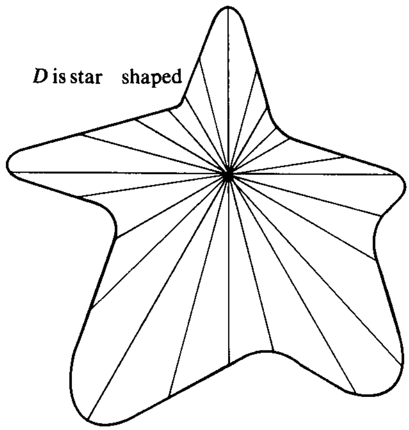


Figure 7.4

Then, differentiating under the integral sign:

$$\frac{\partial f}{\partial x^j}(\mathbf{p}) = \int_0^1 \left[\sum_{i=1}^{n} \frac{\partial a_i}{\partial x^j}(t\mathbf{p})tx^i + a_i(t\mathbf{p})\frac{\partial x^i}{\partial x^j}\right] dt$$

$$= \int_0^1 \sum \frac{\partial a_i}{\partial x^j}(t\mathbf{p})tx^i\, dt + \int_0^1 a_j(t\mathbf{p})\, dt$$

Now, using the compatibility equations, the integrand of the first integral takes

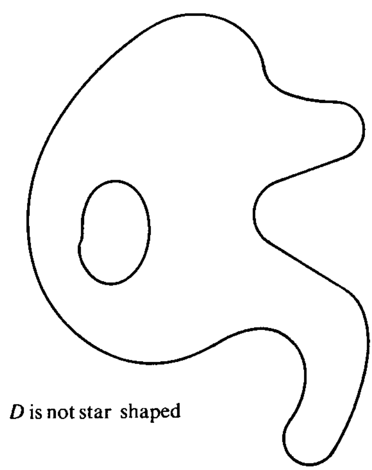


Figure 7.5

the form

$$\sum_i \frac{\partial a_i}{\partial x^j}(t\mathbf{p})tx^i = \left[\sum_i \frac{\partial a_j}{\partial x^i}(t\mathbf{p})x^i\right] \cdot t = \frac{\partial}{\partial t}[a_j(t\mathbf{p})] \cdot t$$

We can now compute the first integral by integration by parts:

$$\int_0^1 \frac{\partial}{\partial t}[a_j(t\mathbf{p})]t\, dt = a_j(t\mathbf{p}) \cdot t \Big|_0^1 - \int_0^1 a_j(t\mathbf{p})\, dt$$

Thus

$$\frac{\partial f}{\partial x_j}(\mathbf{p}) = a_j(\mathbf{p}) \cdot 1 - \int_0^1 a_j(t\mathbf{p})\, dt + \int_0^1 a_j(t\mathbf{p})\, dt = a_j(\mathbf{p})$$

and the proof of Poincaré's lemma is concluded.

Poincaré's lemma serves to indicate the nature of the solution to the basic question: when are closed forms exact? *It depends on the shape of the domain.* If the domain is a ball, or a cube, or any "star-shaped" domain, then every differential form which satisfies the compatibility differential equations (7.38) is the differential of a function. On the other hand, if the domain has holes (as does $R^2 - \{\mathbf{0}\}$), there are closed forms which are not exact. We have seen, to be precise, in the discussion following Example 18 that on $R^2 - \{\mathbf{0}\}$ the dimension of the space of closed forms exceeds the dimension of the space of exact forms by one. Problems 15 and 33 are devoted to showing that when we remove a finite number of points from R^2 this excess dimension on the remaining space is the same as the number of removed points. These examples suggest that domains with holes are not just defective in the closed-exact problem, but further that the solution to this problem *gives a measure of the defect.* This striking relationship between the shape, or topology, of the domain and the analytic question of integrability persists when we move to more complicated domains, or surfaces and even into higher dimensions. The shape of a pretzel is accurately reflected in the closed vs. exact controversy on its surface. The general theorem relating this analysis to the topology of the domain is *de Rham's theorem* and is one of the cornerstones of the modern subject of differential topology.

Now, back in one dimension, the fundamental theorem of calculus relates the values of a function on the boundary of an interval with the integral of its derivative over the interval:

$$f(b) - f(a) = \int_a^b df = \int_a^b \frac{df}{dt}\, dt \tag{7.40}$$

The analog of this theorem for differential forms in R^2 is **Green's theorem**; there are many analogs in higher dimensions and we shall study some of these in the next chapter. For the remainder of the present chapter we shall study only the two-variable case.

Suppose D is a domain in R^2, and the boundary of D is made up of a finite collection of curves (see Figure 7.6). We make the boundary into an oriented path by choosing the direction of motion so that the domain D is always on the left. If $\mathbf{T} \to \mathbf{N}$ is the (right-handed) tangent-normal frame on the domain, then the normal $\mathbf{N}$ always points into the domain (see Figure 7.7). We shall refer to the boundary of D when so oriented as ∂D. Now Green's theorem simply says this: if ω is a C^1 differential form defined on a neighborhood of D, then

$$\int_{\partial D} \omega = \int_D d\omega \tag{7.41}$$

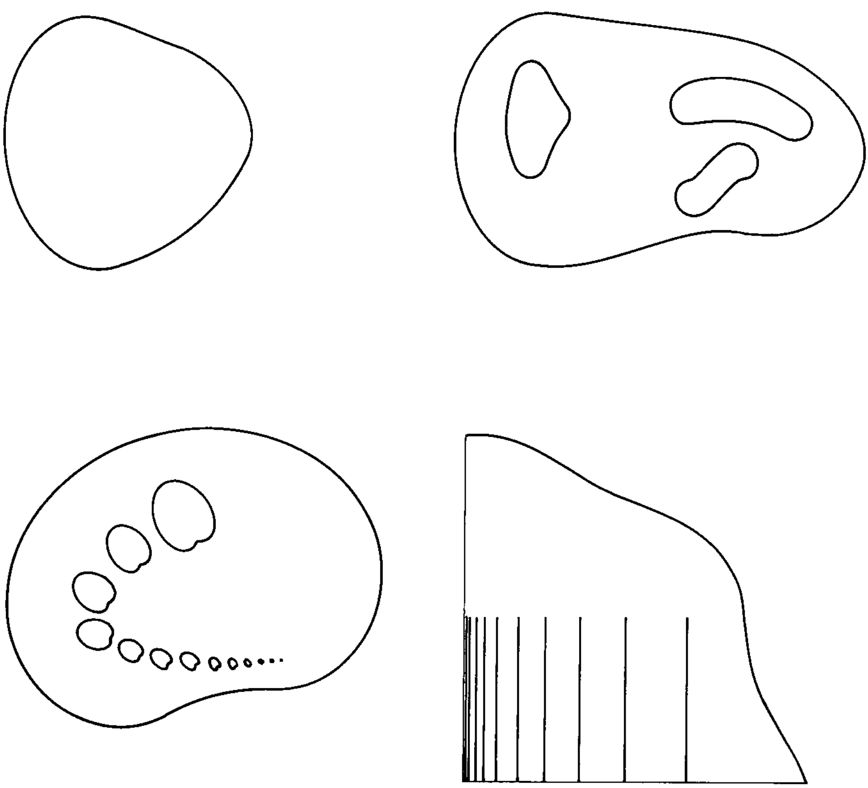


Figure 7.6

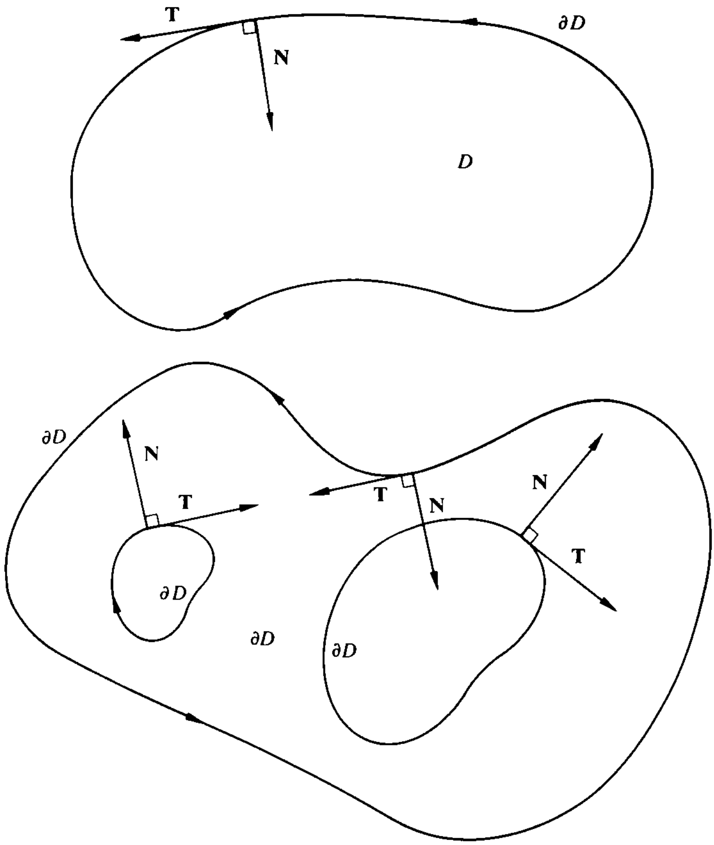


Figure 7.7

If we consider the boundary of the interval in (7.40) as oriented in some appropriate way, then (7.41) appears to be a direct generalization of (7.40).

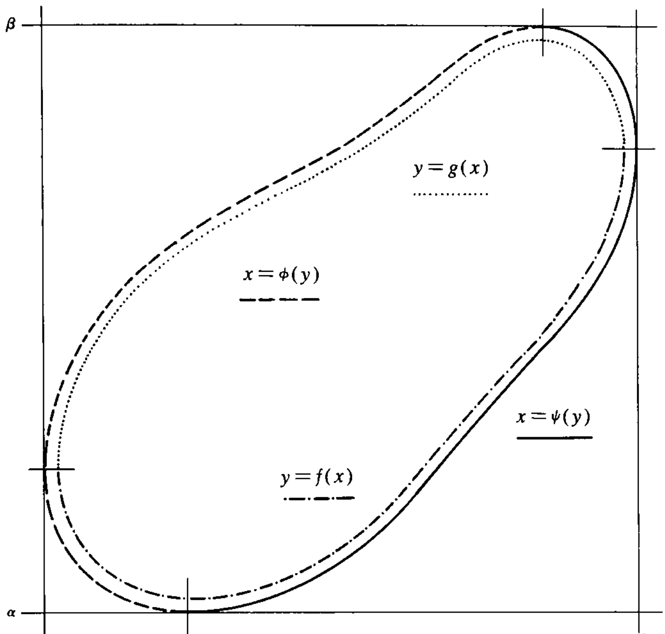
In order to see why (7.41) is true, we must first assume that D is of a special form. We say that the domain D is *regular* if it can be expressed in both following forms:

$$D = \{(x, y) \in R^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\} \tag{7.42}$$

$$= \{(x, y) \in R^n : \alpha \leq y \leq \beta, \phi(y) \leq x \leq \psi(y)\} \tag{7.43}$$

(see Figure 7.8).

For regular domains, Green's theorem follows easily from the fundamental theorem of calculus. Let ω be a given C^1 form, and write $\omega = p\,dx + q\,dy$.



a regular domain

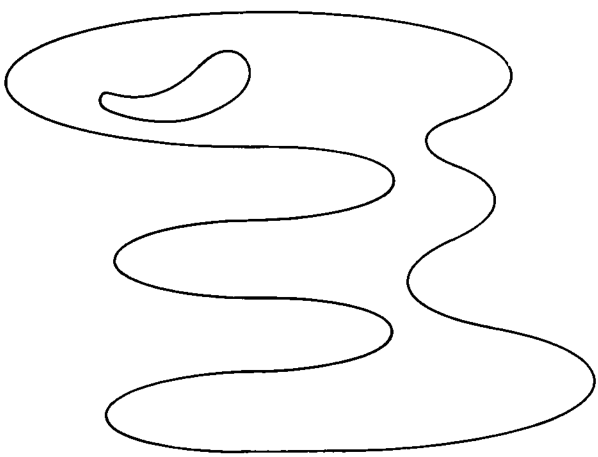

an irregular domain

Figure 7.8

Then

$$\int_D d\omega = \int_D (q_x - p_y)\,dx\,dy = \int_D q_x\,dx\,dy - \int_D p_y\,dy\,dx$$

We perform these integrations, by iteration.: use x first for the first integral, y first for the second.

$$\int_D q_x\,dx\,dy = \int_\alpha^\beta \left[\int_{\phi(y)}^{\psi(y)} \frac{\partial q}{\partial x}\,dx\right] dy = \int_\alpha^\beta q(\psi(y), y)\,dy - \int_\alpha^\beta q(\phi(y), y)\,dy \tag{7.44}$$

Now, we can parametrize ∂D in two parts as:

$$\begin{aligned} &\partial D = \Gamma_1 + \Gamma_2 \\ &\Gamma_1 : \mathbf{x}(t) = (\psi(t), t) \qquad \alpha \le t \le \beta \\ -&\Gamma_2 : \mathbf{x}(t) = (\phi(t), t) \qquad \alpha \le t \le \beta \end{aligned}$$

Thus

$$\int_{\partial D} q\,dy = \int_{\Gamma_1} q\,dy - \int_{-\Gamma_2} q\,dy = \int_\alpha^\beta q(\psi(t). t)\,dt - \int_\alpha^\beta q(\phi(t), t)\,dt \tag{7.45}$$

Comparing (7.44) and (7.45) we deduce that

$$\int_D q_x\,dx\,dy = \int_{\partial D} q\,dy \tag{7.46}$$

We leave it to the reader to verify by the same kind of argument that

$$\int_D p_y\,dx\,dy = -\int_{\partial D} p\,dx \tag{7.47}$$

(Problem 25). Equations (7.46) and (7.47) together give Green's theorem.

Now, not every domain can be represented in both the required ways; in fact, in general neither is possible. However, for most domains D it is true that D can be covered by finitely many disks $\Delta_1, \ldots, \Delta_s$ so that $D \cap \Delta_i = D_i$ is regular for every i. Clearly, if D is bounded by finitely many polygonal curves this is true. All but the most pathological domains that we have seen have this property. The above argument generalizes easily to these types of domains. We shall now call any such domain **regular**.

Definition 9. A domain D is **regular** if its boundary is a path and if D can be covered by disks $\Delta_1, \ldots, \Delta_s$ such that each $D \cap \Delta_i$ can be represented in both forms (7.42) and (7.43).

Theorem 7.6. (Green's Theorem) *Let D be a regular domain and ω a differential form defined on a neighborhood of D. Then,*

$$\int_{\partial D} \omega = \int_D d\omega$$

Proof. Let $D_i \cap \Delta_i$ where the disks $\Delta_1, \ldots, \Delta_n$ are as given in the definition. In particular, by the preceding arguments, Green's theorem is true on D_i for each i. Let $\rho_1, \ldots, \rho_s$ be a partition of unity subordinate to the covering $\Delta_1, \ldots, \Delta_s$. Then the ρ_i are C^∞ functions and $\sum \rho_i = 1$ on D, and ρ_i is nonzero only inside Δ_i. Now, by Green's theorem on D_i

$$\int_{\partial D_i} \rho_i \omega = \int_{D_i} d(\rho_i \omega)$$

Since $\rho_i \omega$ is zero off D_i,

$$\int_{D_i} d(\rho_i \omega) = \int_D d(\rho_i \omega)$$

But also $\rho_i \omega$ is nonzero only on the part of each of the curves ∂D, ∂D_i which is common to both, thus also

$$\int_{\partial D_i} \rho_i \omega = \int_{\partial D} \rho_i \omega$$

Thus

$$\int_{\partial D} \rho_i \omega = \int_D d(\rho_i \omega) \qquad 1 \le i \le s$$

Adding these equations, we obtain Green's theorem for D since $\sum \rho_i = 1$:

$$\int_{\partial D} \omega = \int_{\partial D} \sum \rho_i \omega = \sum \int_{\partial D} \rho_i \omega = \sum \int_D d(\rho_i \omega) = \int_D d(\sum \rho_i \omega) = \int_D d\omega$$

Examples

25. Let D be the unit rectangle $[(0, 0), (1, 1)]$. Then, by Green's theorem

$$\int_{\partial D} x^2 y \, dx + (x - y) \, dy = \int_D (1 - x^2) \, dx \, dy = \int_0^1 (1 - x^2) \, dx = \frac{2}{3}$$

26. The integral of $\omega = \cos xy\, dx + y \cos x\, dy$ over the boundary of the domain

$$D = \{(x, y) : 0 \leq x^2 \leq y \leq 1\}$$

is

$$\int_{\partial D} \omega = \int_D [-y \sin x + x \cos xy]\, dx\, dy$$
$$= \int_{-1}^{1} \left[\int_{x^2}^{1} [(-y \sin x) + x \cos xy]\, dy\right] dx$$

Green's theorem is also convenient for transforming double integrals into line integrals. Noticing that $dx\, dy$ arises as $d(-y\, dx)$ or $d(x\, dy)$ in Green's theorem, we may compute areas of domains by line integrals.

27. Find the area bounded by the curves $y = 1 - x^4$ and $y = 1 - x^6$ in the upper half plane:

$$\text{area} = \int_D dx\, dy$$
$$= -\int_{\partial D} y\, dx = -\int_{-1}^{1} (1 - x^4)\, dx + \int_{-1}^{1} (1 - x^6)\, dx = \frac{4}{35}$$

28. Find the area inside the ellipse

$$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We can parametrize E by the polar angle:

$$x = a \cos \theta \qquad y = b \sin \theta$$

Thus

$$\text{area} = \int_E x\, dy = ab \int_0^{2\pi} \cos^2 \theta\, d\theta = \pi ab$$

• EXERCISES

14. Compute the line integrals of differential forms arising out of the work problems in Exercise 12(a), (b) using Green's theorem.

15. Compute $\int_\Gamma \omega$ for given ω and Γ (using Green's theorem if convenient).

(a) $\omega = z\,dx + x\,dy + y\,dz$ $\quad\Gamma$: closed oriented polygon with successive vertices $(0, 0, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(-1, -1, -1)$.

(b) $\omega = x^2y\,dx + y^2x\,dy$ $\quad\Gamma$: the ellipse $a^2x^2 + b^2y^2 = 1$.

(c) $\omega = (x + y)\,dx + (x^2 + y^2)\,dy$ $\quad\Gamma$: the triangle with successive vertices $(0, 0)$, $(4, 0)$, $(2, 3)$.

(d) $\omega = x^2\,dy + 2xy\,dx$ $\quad\Gamma$: $z = e^{(1+i)t}$ from $t = 0$ to $t = 2$.

(e) $\omega = (x + y)\,dx + (y + z)\,dy + (z + x)\,dx$
Γ: the circle $x^2 + z^2 = 1$, $y = 3$.

16. Compute, using Green's theorem the area of the domain D:

(a) $D = \{(x, y):\quad 0 \le \sin x \le y \le \tan x \le 1\}$

(b) D is the domain in the upper half plane bounded by the ellipse $x^2 + 2y^2 = 1$ and the parabola $x = 2y^2$.

(c) D is the quadrilateral with vertices at $(0, 0)$, $(1, 0)$, $(7, 3)$, $(2, 5)$.

(d) Inside the curve $x = \cos^n t$, $y = \sin^n t$ $\quad n > 0$.

• PROBLEMS

25. Verify Equation (7.47) in the text and conclude the proof of Green's theorem.

26. Using Green's theorem prove that if ω is a closed differential form in all of R^2, then ω is exact.

27. A differential form is called **radial**, if it is of the form $\langle \mathbf{F}, d\mathbf{x}\rangle$ where $\mathbf{F}$ is a radial vector field (see Problem 24). Show that if ω is radial, it is of the form $f(r)\,dr$.

28. Show that if ω is a compactly supported (that is, it is identically zero outside some large disk) form on the plane that

(a) $\displaystyle\int_{R^2} d\omega = 0$

(b) $\displaystyle\int_{x\text{ axis}} \omega = \int_{y>0} d\omega$

29. Show that if ω is a compactly supported closed form in R^2, it is the differential of a compactly supported function.

30. If ω is a differential form, define $*\omega$ as follows: if

$$\omega = \langle \mathbf{F}, d\mathbf{x}\rangle \qquad *\omega = \langle *\mathbf{F}, d\mathbf{x}\rangle$$

(a) Show that if $\omega = p\,dx + q\,dy$, $*\omega = -q\,dx + p\,dy$.

(b) Show that (in a disk) $*df$ is also exact if and only if f is harmonic.

(c) Show, using complex notation

$$*\omega(\mathbf{T}) = \omega(i\mathbf{T})$$

(d) If f is harmonic, let f^* be such that $df^* = {*df}$. Show that $f + if^*$ satisfies the Cauchy–Riemann equations.

31. Let Γ be an oriented curve in R^n, with tangent $\mathbf{T}$ and normal $\mathbf{N}$. If f is a differentiable function we define these derivatives of f along Γ:

$$\frac{\partial f}{\partial \mathbf{T}} = df(\mathbf{T}) = \langle \nabla f, \mathbf{T} \rangle \qquad \frac{\partial f}{\partial \mathbf{N}} = df(\mathbf{N}) = \langle \nabla f, \mathbf{N} \rangle$$

Show that

$$\int_\Gamma df = \int_\Gamma \frac{\partial f}{\partial \mathbf{T}}\, ds \qquad \int_\Gamma \frac{\partial f}{\partial \mathbf{N}}\, ds = \int_\Gamma {*df}$$

32. Suppose that D is a regular domain and f, g are twice differentiable functions defined on a neighborhood of D. Verify these formulas (using Green's theorem):

(a) $$\int_{\partial D} \frac{\partial f}{\partial \mathbf{T}}\, ds = 0$$

(b) $$\int_{\partial D} g \frac{\partial f}{\partial \mathbf{T}}\, ds = \iint_D \left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x} \right) dx\, dy$$

(c) $$\int_{\partial D} \left(g \frac{\partial f}{\partial \mathbf{T}} + f \frac{\partial g}{\partial T} \right) ds = 0$$

(d) $$\int_{\partial D} \frac{\partial f}{\partial \mathbf{N}}\, ds = \iint_D \Delta f\, dx\, dy$$

(e) $$\int_{\partial D} g \frac{\partial f}{\partial \mathbf{N}}\, ds = \iint_D [g\Delta f + \langle \nabla g, \nabla f \rangle]\, dx\, dy$$

(f) $$\int_{\partial D} \left(g \frac{\partial f}{\partial \mathbf{N}} - f \frac{\partial g}{\partial \mathbf{N}} \right) ds = \iint_D (g\Delta f - f\Delta g)\, ds$$

7.6 Applications of Green's Theorem

Several of the exercises at the end of the previous section have indicated the uses of Green's theorem. The rest of this chapter is devoted to the *application* of this theorem to some of the topics we have been developing. We shall leave aside until the next section its more profound uses in the study of complex differentiable functions.

The Shape of the Domain

The most immediate implication of Green's theorem is the suggestion of the relationship of the shape of a domain to the question of the exactness of closed forms. If every closed curve in the domain D is the boundary of a subdomain in D, then every closed form is exact. For, suppose ω is a closed form. By Theorem 7.4 (i), to show that ω is exact, we need only verify that its integral over any closed curve is zero. If Γ is such a curve, then by hypothesis it is the boundary of the subdomain E. Then, by Green's theorem

$$\int_\Gamma \omega = \int_E d\omega = 0$$

since $d\omega = 0$.

We can say that a domain D "has no holes" if every closed curve in D is the boundary of a subdomain of D. This is intuitively clear: we can draw a loop around any hole which will bound the hole and this is not a subdomain in D. The further study and precision of these notions is a rather difficult branch of mathematics and falls within the domain of *topology*. It turns out that there is a precise relation between this vague geometric study and the question of exactness. The number of "holes" in the domain is the same as the number of independent closed but nonexact forms. We already saw that (in Section 7.2) for $R^2 - \{0\}$ and in Problem 15 for $R^2 - \{0, 1\}$. That argument easily generalizes to the case of the complement of finitely many points, $p_1, \ldots, p_s$. Let $\theta_i(z) = \arg(z - p_i)$. Although θ_i is not a well-defined function on $R^2 - \{p_1, \ldots, p_s\}$, $d\theta_i$ is a well-defined form. Clearly, $d\theta_1, \ldots, d\theta_s$ are independent, so there are at least s independent closed nonexact forms on $R^2 = \{p_1, \ldots, p_s\}$. Now, let ω be any closed form and define

$$c_i(\omega) = \frac{1}{2\pi i} \int_{C_i} \omega$$

where C_i is a small circle centered at p_i. Then

$$\omega' = \omega - \frac{1}{2\pi} \sum_{i=1}^{s} c_i(\omega)\, d\theta_i$$

is exact. This can be proven by verifying condition (i) of Theorem 7.4 by Green's theorem (see Problem 33). Thus if ω is any closed form it is, but for an exact form, a linear combination of the $d\theta_i$.

Area Computation

Now, as in Examples 27, 28, we can compute areas by boundary integrals: if D is a regular domain

$$\text{area of } D = \iint_D dx\,dy = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx \tag{7.48}$$

Example

29. The area of a trapezoid is $1/2(b_1 + b_2)h$ (see Figure 7.9).

$$\text{area} = \int_{\partial D} x\,dy = \int_{L_1} x\,dy + \int_{L_2} x\,dy$$

$$L_1 : y = \frac{h}{\alpha + b_2 - b_1}(x - b_1) \qquad x \in [\alpha + b_2, b_1]$$

$$L_2 : y = \frac{h}{\alpha}x \qquad x \in [0, \alpha]$$

$$\begin{aligned}\text{area} &= \int_{b_1}^{\alpha+b_2} x\frac{h}{\alpha + b_2 - b_1}\,dx + \int_{\alpha}^{0} x\frac{h}{\alpha}\,dx \\ &= \frac{h}{2}\left[\frac{(\alpha + b_2)^2 - {b_1}^2}{\alpha + b_2 - b_1}\right] - \frac{h}{2\alpha}\alpha^2 \\ &= \frac{h}{2}[\alpha + b_2 + b_1 - \alpha] = \frac{1}{2}(b_1 + b_2)h\end{aligned}$$

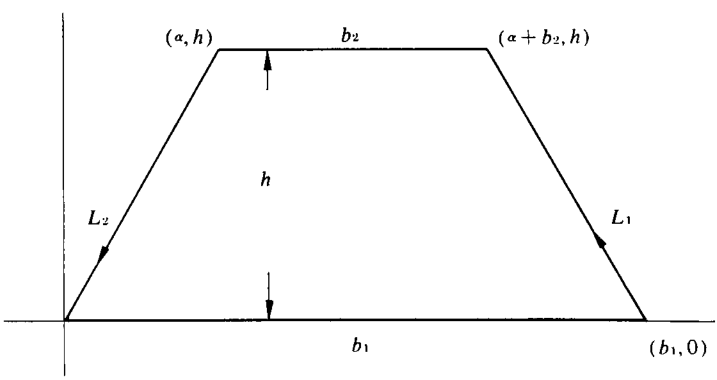


Figure 7.9

Integration after a Change of Variable

A line integral of a differential form is the same, no matter what coordinates are used to compute it (recall Proposition 4). Using this knowledge and the preceding computational techniques we can find a formula for computing double integrals by a coordinate change.

Suppose that **F** is a nonsingular differentiable transformation of the domain D onto the domain E (that is, **F** maps D one-to-one onto E and $d\mathbf{F}$ is everywhere nonsingular). Let us write **F** in terms of coordinates:

$$\mathbf{F}: \begin{matrix} u = u(x, y) \\ v = v(x, y) \end{matrix} \quad (x, y) \in D \qquad \mathbf{F}^{-1}: \begin{matrix} x = x(u, v) \\ y = y(u, v) \end{matrix} \quad (u \ \ v) \in E \tag{7.49}$$

If Γ is a path in D, then $\mathbf{F}(\Gamma)$ is a path in E. If $\omega = p\,dx + q\,dy$ is a differential form defined on D, we may associate it to a form on E: $\tilde{\omega} = \alpha\,du + \beta\,dv$, where the cooefficients are given (see (7.24)) by the coordinate change $(x, y) \to (u, v)$. Then $\int_\Gamma \omega = \int_{F(\Gamma)} \tilde{\omega}$, since they represent the same integration relative to two different coordinate sets. Now, if Γ bounds a domain Δ, $\mathbf{F}(\Gamma)$ bounds $\mathbf{F}(\Delta)$ and if we apply Green's theorem to both sides we will obtain a relation between the double integrals. However, to apply Green's theorem we must be sure that both Γ and $\mathbf{F}(\Gamma)$ are oriented as the boundary of the domains Δ, $\mathbf{F}(\Delta)$, respectively. That is not necessarily the case.

Example

30. The transformation

$$\mathbf{F}: \begin{matrix} u = y \\ v = x \end{matrix}$$

amounts to reflection in the line $x = y$. If Γ is a circle centered on that line, Γ and $\mathbf{F}(\Gamma)$ are the same curve, but oriented in opposite directions (see Figure 7.10).

This difficulty may be overcome by restricting attention exclusively to transformations that preserve the sense of orientation around a curve. This will be guaranteed if the sense of "counterclockwise" rotation about corresponding points is the same. Thus, if we rotate the xy plane about the point **p** in the clockwise sense, the induced motion under the transformation **T** must also be clockwise. This will be the case if it is so for the linear

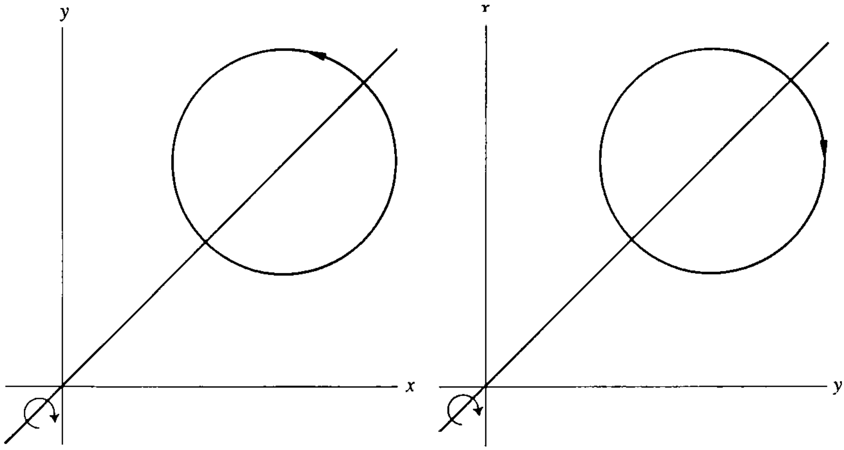


Figure 7.10

approximation $d\mathbf{T}(\mathbf{p})$, and that is guaranteed by

$$\frac{\partial(x, y)}{\partial(u, v)}(\mathbf{p}) = \det\begin{pmatrix} \dfrac{\partial x}{\partial u}(\mathbf{p}) & \dfrac{\partial x}{\partial v}(\mathbf{p}) \\ \dfrac{\partial y}{\partial u}(\mathbf{p}) & \dfrac{\partial y}{\partial v}(\mathbf{p}) \end{pmatrix} > 0 \tag{7.50}$$

These remarks are not completely obvious, but we shall not pause to verify them. It is intuitively clear that the sense of rotation at a point is the same for the transformation and its differential. What is not so clear, and more difficult to obtain is that this local criterion assures that the sense of orientation of any boundary is the same in the two coordinate systems. All these geometric considerations can be avoided, by replacing them with appropriate algebraic considerations. We shall see further illustrations of the difficulty in a purely geometric, rather than algebraic, approach in the next chapter.

In any event, if (7.49) defines a change of variables satisfying condition (7.50), then for any subdomain Δ of D, $\partial\Delta$ and $\partial\mathbf{F}^{-1}(\Delta)$ define the same orientation on the boundary of ω. Thus, if ω is any differential form

$$\int_{\partial\Delta} \omega = \int_{\partial F^{-1}(\Delta)} \omega$$

In particular,

$$\text{area}(\Delta) = \int_{\partial\Delta} x\,dy = \int_{\partial F^{-1}(\Delta)} x\,dy = \int_{\partial F^{-1}(\Delta)} x\,\frac{\partial y}{\partial u}\,du + x\frac{\partial y}{\partial v}\,dv$$

$$= \int_{F^{-1}(\Delta)} \left[\frac{\partial}{\partial u}\left(x\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(x\frac{\partial y}{\partial u}\right)\right] du\,dv$$

$$= \int_{F^{-1}(\Delta)} \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv$$

A more important formula is that allowing us to compute double integrals with respect to the new coordinates (u, v).

Theorem 7.7. *Let D be a domain in the plane, and suppose*

$$x = x(u, v)$$
$$y = y(u, v)$$

is an orientation-preserving change of coordinates (that is, $\partial(x, y)/\partial(u, v) > 0$). Let E be the domain in (u, v) variables corresponding to D. If f is a function defined on a rectangle containing D, then $\int_D f$ can be computed in terms of the (u, v) coordinates:

$$\int_D f = \int_E f(x(u, v), y(u, v))\det\frac{\partial(x, y)}{\partial(u, v)}(u, v)\,du\,dv \tag{7.51}$$

Proof. Let $R = [(a, b), (\alpha, \beta)]$ and define

$$F(x, y) = \int_a^x f(t, y)\,dt \qquad \text{for } (x, y) \in R$$

Thus $F(x, y)$ is a C^1 differentiable function on R such that $\partial F/\partial x = f$. Now, by Green's theorem

$$\int_D f\,dx\,dy = \int_{\partial D} F\,dy$$

We can compute the integral over ∂D in the (u, v) coordinates:

$$\int_{\partial D} F\,dy = \int_{\partial E} F\,dy = \int_{\partial E} F\frac{\partial y}{\partial u}\,du + F\frac{\partial y}{\partial v}\,dv$$

By Green's theorem (in the (u, v) variables), the last integral is

$$\int_E \left[\frac{\partial}{\partial u}\left(F\frac{\partial y}{\partial v}\right) - \frac{\partial}{\partial v}\left(F\frac{\partial y}{\partial u}\right)\right] du\, dv$$

$$= \int_E \left[\frac{\partial F}{\partial x}\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right.$$

$$\left. + F\frac{\partial^2 y}{du\, dv} - \frac{\partial F}{\partial x}\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial y}{\partial v}\frac{\partial y}{\partial u} - F\frac{\partial^2 y}{dv\, du}\right] du\, dv$$

$$= \int_E f(x(u, v), y(u, v)) \det \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv$$

Thus (7.51) is proven.

Examples

31.

$$\int_{x^2+y^2\le 1} \frac{dx\, dy}{(x^2+y^2)^{1/2}} = \int_{x^2+y^2\le 1} \frac{r\, dr\, d\theta}{r} = \int_0^1 \left[\int_0^{2\pi} d\theta\right] dr = 2\pi$$

32.

$$\int_{R^2} \exp[-(x^2+y^2)]\, dx\, dy = \int_{R^2} \exp(-r^2) r\, dr\, d\theta$$

$$= 2\pi \int_0^\infty \exp(-r^2) r\, dr$$

$$= \pi[-\exp(-r^2)]_0^\infty = \pi$$

Notice that

$$\left(\int_0^\infty \exp(-t^2)\, dt\right)^2 = \int_0^\infty \exp(-t^2)\, dt \cdot \int_0^\infty \exp(-t^2)\, dt$$

$$= \int_0^\infty \exp(-x^2)\, dx \cdot \int_0^\infty \exp(-y^2)\, dy = \int_{R^2} \exp[-(x^2+y^2)]\, dx\, dy$$

Thus

$$\int_0^\infty \exp(-t^2)\, dt = \sqrt{\pi}$$

a computation that would have been impossible without the change of variable to polar coordinates.

The Divergence Theorem

The general form of Green's theorem first came up in the study of fluid flows and the theory of potentials. In this study it arises in the form of the **divergence theorem**, which we shall now discuss in two variables.

Let $\mathbf{v} = (v, w)$ be a vector field defined in some open set in the plane and let $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ be the equations of the associated flow (that flow with velocity field $\mathbf{v}$). Let D be a domain on which the flow takes place. The fluid which at time $t = 0$ occupies D has moved after a time t, to a domain D_t given by

$$D_t = \{\mathbf{x}: \mathbf{x} = \mathbf{x}(\mathbf{x}_0, t) \qquad \mathbf{x}_0 \in D\}$$

The area of D_t is

$$\text{area}(D_t) = \int_{D_t} dx\, dy = \int_D \frac{\partial(x_t, y_t)}{\partial(x_0, y_0)}\, dx_0\, dy_0$$

where we have rewritten the equations of flow as

$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t) = (x_t(x_0, y_0), y_t(x_0, y_0))$$

The rate of change of the area of D_t is

$$\frac{d}{dt}\,\text{area}(D_t) = \int_D \frac{\partial}{\partial t}\left[\frac{\partial(x_t, y_t)}{\partial(x_0, y_0)}\right] dx_0\, dy_0 \tag{7.52}$$

Now let us evaluate this at time $t = 0$. Remembering that $\mathbf{x}(\mathbf{x}_0, 0) = \mathbf{x}_0$, we have

$$\frac{\partial}{\partial t}\left[\frac{\partial x_t}{\partial x_0}\frac{\partial y_t}{\partial y_0} - \frac{\partial x_t}{\partial y_0}\frac{\partial y_t}{\partial x_0}\right]\Bigg|_{t=0} = \frac{\partial^2 x}{\partial t\, \partial x_0}(x_0, y_0\ \ 0) + \frac{\partial^2 y}{\partial t\, \partial y_0}(x_0, y_0, 0)$$

Now

$$\frac{\partial^2 x}{\partial t\, \partial x_0} = \frac{\partial}{\partial x_0}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial v}{\partial x_0} \qquad \frac{\partial^2 y}{\partial t\, \partial y_0} = \frac{\partial w}{\partial y_0}$$

Thus the instantaneous rate of change of the area of D (Equation (7.52)) is given by

$$\int_D \left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\right) dx\, dy \tag{7.53}$$

The integrand is called the **divergence of the flow** and is denoted div $\mathbf{v}$. The divergence theorem says that this integral can be computed by a boundary integral. To put it physically: the rate of expansion of D is the same as the rate at which fluid flows into D. We will now try to compute that latter amount. Let ∂D have the frame $\mathbf{T} \to \mathbf{N}$ so that $\mathbf{N}$ points into the domain (see Figure 7.11). The amount of fluid passing into ∂D through a small piece of the boundary (of length Δs) in a time Δt is

$$\langle \mathbf{v}, \mathbf{N} \rangle \, \Delta s \, \Delta t$$

The total amount passing through ∂D is thus well approximated by a Riemann sum for the integral

$$\left(\int_{\partial D} \langle \mathbf{v}, \mathbf{N} \rangle \, ds \right) \Delta t$$

Thus the rate at which fluid passes into D can be thought to be given by

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{N} \rangle \, ds$$

Using the notation of Exercise 29 this is the same as

$$\int_{\partial D} \langle {}^*\mathbf{v}, d\mathbf{x} \rangle = \int_{\partial D} -w \, dx + v \, dy$$

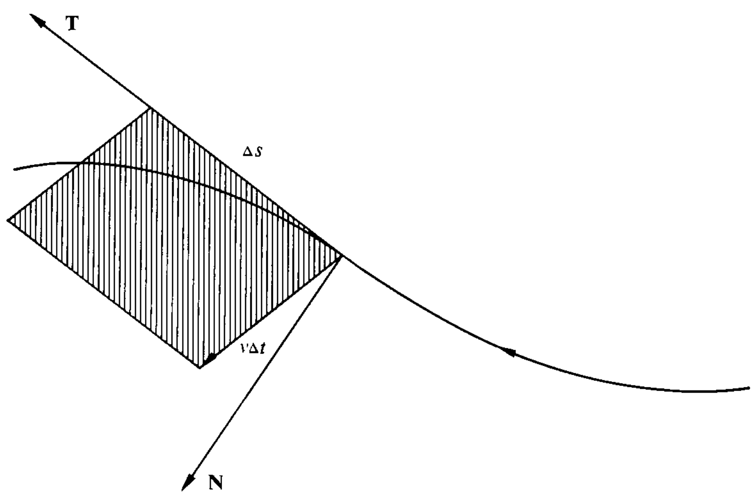


Figure 7.11

By Green's theorem this is the same as (7.53). Thus the divergence theorem is verified:

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{N} \rangle \, ds = \int_D \operatorname{div} \mathbf{v} \tag{7.54}$$

If $\mathbf{v}$ is a conservative field it has a potential function f, and $\langle \mathbf{v}, d\mathbf{x} \rangle = df$. Then $\langle {}^*\mathbf{v}, d\mathbf{x} \rangle = {}^*df$ and (7.54) becomes

$$\int_{\partial D} {}^*df = \int_D d^* \, df = \int_D \Delta f$$

Thus, if f is the potential function for a conservative and incompressible (divergence free) flow, f must be a harmonic function. Dirichlet's problem (to find a harmonic function with given boundary values) may be restated as: find the conservative incompressible flow with given boundary potential levels.

The Cauchy Theorem

This last remark leads directly to the study of complex analysis. Suppose that f is a complex-valued complex differentiable C^1 function defined on a domain in the plane. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

exists for all z and (what is the same assertion) the Cauchy-Riemann equations hold:

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

It follows that the form $f(z)\, dz$ is a closed complex-valued form:

$$f\, dz = f\, dx + if\, dy$$

$$d(f\, dz) = \left[\frac{\partial}{\partial x}(if) - \frac{\partial f}{\partial y}\right] = if_x - f_y = 0$$

Theorem 7.8. (Cauchy's Theorem) *If f is a C^1 complex differentiable function defined in the regular domain D, then*

$$\int_{\partial D} f\, dz = 0$$

Proof. By Green's theorem

$$\int_{\partial D} f\,dz = \int_D d(f\,dz) = 0$$

• EXERCISES

17. Compute the area of these domains:
    (a) $x^4 + y^4 \leq a^4$
    (b) $x^2 y \leq 1, 0 \leq x \leq a$
    (c) $r \leq 1 + 2\cos\theta$ (each section)
    (d) $r \leq e^\theta, 0 \leq \theta \leq 2\pi$
    (e) The domain $\{u^2 + v^2 \leq 1/2\}$, where

$$u = x(1 + x\cos v)$$
$$v = y(1 + y\cos x)$$

    (f) The domain $\{0 \leq u \leq 1,\ 0 \leq v \leq 1\}$, where $u = xy$, $v = x^2 - y^2$.
18. Compute div $\mathbf{v}$ for these flows:
    (a) $\mathbf{x}(\mathbf{x}_0, t) = \exp\left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} t\right] \mathbf{x}_0$
    (b) $x = x_0(1 + t), y = y_0(1 - t^2)$
    (c) $\mathbf{v}(x, y) = (x^2 - y, y^2 - x)$
    (d) $\mathbf{v}(x, y) = (x + y, x - y)$

• PROBLEMS

33. Let $D = R^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_s\}$, where $\mathbf{p}_1, \ldots, \mathbf{p}_s$ are s distinct points in the plane. Show that there is an s-dimensional space L of closed, but not exact forms defined on D such that every closed form can be written $df + \omega$, with $\omega \in L$.

34. Let ω be a closed form in $R^2 - \{(0, 0)\}$. Show that if ω is exact in some annulus $\{a \leq |z| \leq b\}$, then it is exact.

35. Let f be a complex-valued differentiable function defined in the domain E. Show that f is complex analytic if and only if $\int_{\partial D} f\,d_z = 0$ for all subdomains D of E. (*Hint*: $d(f\,dz) = 0$ is the same as the Cauchy–Riemann equations.)

7.7 The Cauchy Integral Formula

In Chapter 5 we introduced the power series development of functions in order to effectively compute solutions to certain differential equations. Those functions which admit an expansion into a power series are called **analytic.** We saw that this is the most computable class of functions. We

saw that such functions are differentiable in the complex sense, and that the differential equations can be interpreted in the sense of complex variables. In Chapter 6 we found that if a function is the sum of a convergent power series in the closed unit disk, it can be computed by means of an integral around the circle:
if

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \text{ for } |\zeta| \leq 1$$

then

$$f(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})e^{i\theta}}{e^{i\theta} - \zeta} \, d\theta$$

for $|\zeta| < 1$. The integral may be rewritten as a line integral:

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)\, dz}{z - \zeta}$$

The Cauchy integral formula is a great generalization of this. It weakens the hypothesis to that of complex differentiability and strengthens the conclusion by replacing the unit circle by the boundary of any regular domain.

Theorem 7.9. (Cauchy Integral Formula) *Suppose that f is a C^1 complex-valued complex differentiable function defined in a neighborhood of the regular domain D. Then, for $\zeta \in D$,*

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\, dz}{z - \zeta} \tag{7.55}$$

Proof. Let $\Delta_n = \{z\colon |z - \zeta| < n^{-1}\}$. If n is large enough, Δ_n is contained in D (see Figure 7.12) and $f(z)(z - \zeta)^{-1}$ is a complex differentiable in $D - \Delta_n$. This is because the product of complex differentiable functions is complex differentiable. Thus $f(z)(z - \zeta)^{-1}\, dz$ is closed, so that

$$\int_{\partial(D-\Delta_n)} \frac{f(z)\, dz}{z - \zeta} = 0$$

Thus

$$\int_{\partial D} \frac{f(z)\, dz}{z - \zeta} = \int_{\partial \Delta_n} \frac{f(z)\, dz}{z - \zeta} = i\int_0^{2\pi} \frac{f(\zeta + n^{-1}e^{i\theta})}{n^{-1}e^{i\theta}} n^{-1}\, e^{i\theta}\, d\theta = i\int_0^{2\pi} f(\zeta + n^{-1}e^{i\theta})\, d\theta$$

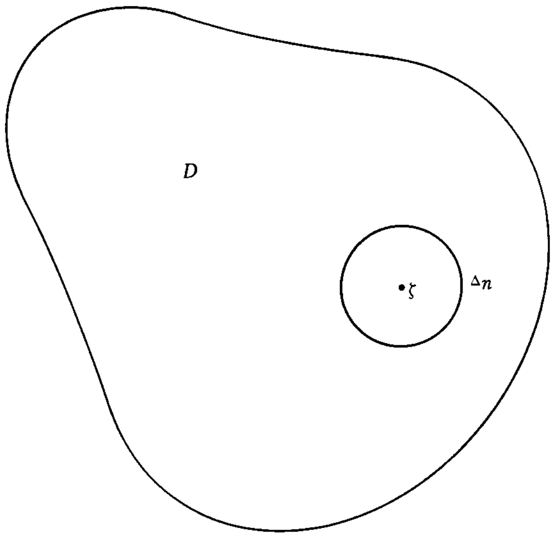


Figure 7.12

But as $n \to \infty$, $f(\zeta + n^{-1}e^{i\theta}) \to f(\zeta)$ uniformly on the circle, just because f is continuous at ζ. Since n is arbitrary (but large),

$$\int_{\partial D} \frac{f(z)\,dz}{z-\zeta} = \lim i \int_0^{2\pi} f(\zeta + n^{-1}e^{i\theta})\,d\theta = i \int_0^{2\pi} f(\zeta)\,d\theta = 2\pi i f(\zeta)$$

and thus (7.55) is proven.

The Cauchy integral formula implies that complex differentiable functions are extremely well behaved; after all a function certainly must be quite special for it to be completely and explicitly determined within a domain by its boundary values. Here are a few corollaries of Theorem 7.9 which demonstrate this.

For simplicity of notation we shall write $f \in \Delta(D)$ to mean that f is a C^1 complex differentiable function on a regular domain D.

Proposition 5. (The Maximum Principle) Let f be in $\Delta(D)$. The maximum of f on D is attained on ∂D.

Proof. Since D is compact, the maximum of f is attained at some point $\zeta \in D$. If there is no point on ∂D at which f attains its maximum, then not only is $\zeta \notin \partial D$, but

$$|f(\zeta)| > \max\{|f(z)| : z \in \partial D\}$$

We shall show that this assumption leads to a contradiction. Define

$$g(z) = \frac{f(z)}{f(\zeta)}$$

Then $g \in \Delta(D)$ also, $g(\zeta) = 1$ and $\|g\|_{\partial D} < 1$. Then $g^n \to 0$ uniformly on ∂D as $n \to \infty$. Thus

$$\int_{\partial D} \frac{g^n(z)\,dz}{z - \zeta} \to 0$$

as $n \to \infty$. But, by the Cauchy integral formula, that integral is $2\pi i g^n(\zeta) = 2\pi i$ which does *not* tend to zero.

Proposition 6. *Suppose* f_n, f *are all in* $\Delta(D)$ *and* $\lim f_n = f$ *uniformly on* ∂D. *Then* $\lim f_n = f$ *uniformly in* D.

Proof. By assumption, $\|f_n - f\|_{\partial D} \to 0$ as $n \to \infty$. But since $f_n - f \in \Delta(D)$, by the maximum principle,

$$\|f_n - f\|_D = \|f_n - f\|_{\partial D} \quad \text{so } \|f_n - f\|_D \to 0 \quad \text{as } n \to \infty \text{ also}$$

Proposition 7. (Liouville's Theorem) *If* f *is bounded and complex differentiable on the entire plane,* f *is constant.*

Proof. Let M be an upper bound for $|f(z)|$. Let ζ_1, ζ_2 be any two points on the plane.

$$\begin{aligned}
|f(\zeta_1) - f(\zeta_2)| &= \left| \frac{1}{2\pi i} \int_{|z|=R} \left[\frac{1}{z - \zeta_1} - \frac{1}{z - \zeta_2} \right] f(z)\,dz \right| \\
&\le \frac{M}{2\pi} |\zeta_1 - \zeta_2| \int_{|z|=R} \left| \frac{dz}{(z - \zeta_1)(z - \zeta_2)} \right| \\
&\le \frac{MR}{2\pi} |\zeta_1 - \zeta_2| \int_0^{2\pi} \frac{d\theta}{(Re^{i\theta} - \zeta_1)(Re^{i\theta} - \zeta_2)} \\
&\le \frac{M}{2\pi R} |\zeta_1 - \zeta_2| \int_0^{2\pi} \frac{d\theta}{|e^{i\theta} - \zeta_1 R^{-1}|\,|e^{i\theta} - \zeta_2 R^{-1}|}
\end{aligned}$$

As $R \to \infty$, the integrand converges to 1. Thus the entire expression on the right becomes arbitrarily small as $R \to \infty$. On the other hand, the left-hand side is independent of R, hence must be zero. Thus, $f(\zeta_1) = f(\zeta_2)$ for any ζ_1, ζ_2.

The most important property of complex differentiable functions is that they are analytic, that is, they can be expressed as the sum of a convergent power series about any point. The following theorem brings together all the notions of analyticity and summarizes the basic properties of analytic functions.

Theorem 7.10. *Let f be a C^1 complex-valued function defined in a neighborhood of the regular domain D. The following assertions are equivalent:*

(i) *For any $\zeta \in D$, and R such that the disk $\Delta(\zeta, R)$ is contained in D, f is the sum in $\Delta(\zeta, R)$ of a convergent power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\zeta)^n \tag{7.56}$$

(ii) *f is complex differentiable.*
(iii) *f satisfies the Cauchy–Riemann equations:*

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

(iv) *$f\,dz$ is closed.*
(v) *for any $\zeta \in D$,*

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)\,dz}{z-\zeta}$$

In case f has these properties the coefficients a_n of (7.56) *are given by*

$$a_n = \frac{f^{(n)}(\zeta)}{n!} = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)\,dz}{(z-\zeta)^{n+1}} \tag{7.57}$$

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) were observed in Chapter 5, (iii) $\Rightarrow$ (iv) in the preceding section and (iv) $\Rightarrow$ (v) is the Cauchy integral formula (Theorem 7.10). That leaves only the implication (v) $\Rightarrow$ (i) and the first part of the theorem will be proven. Suppose then, that (v) holds, and $\Delta(\zeta, R) \subset D$. We have to show that f can be expanded in a power series centered at ζ. By hypothesis, $\min\{|z-\zeta| : z \in \partial D\} \geq R$. Thus for $w \in \Delta(\zeta, R)$,

$$\left|\frac{w-\zeta}{z-\zeta}\right| < 1$$

for all $z \in \partial D$. Thus

$$\frac{1}{z-w} = \frac{1}{z-\zeta-(w-\zeta)} = \frac{1}{z-\zeta}\left(1-\frac{w-\zeta}{z-\zeta}\right)^{-1} = \sum_{n=0}^{\infty} \frac{(w-\zeta)^n}{(z-\zeta)^{n+1}}$$

uniformly for $z \in \partial D$. We can thus substitute this sum for the term $(z-w)^{-1}$ in the Cauchy integral:

$$f(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)\,dz}{z-w} = \frac{1}{2\pi i}\int_{\partial D} f(z) \sum_{n=0}^{\infty} \frac{(w-\zeta)^n}{(z-\zeta)^{n+1}}\,dz$$
$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{\partial D} \frac{f(z)\,dz}{(z-\zeta)^{n+1}}\right](w-\zeta)^n$$

Thus f is represented by a power series whose coefficients are given by the integrals in (7.57). That the coefficients also are given by the successive derivatives as in (7.57) was already observed as part of Taylor's formula. Thus, the theorem is completely proven.

Examples

33. If f is analytic in the disk $\Delta(\zeta, R)$, then the power series representing f near ζ actually converges to f in the entire disk $\Delta(\zeta, R)$. For, by Theorem 7.10, f is, in this whole disk, the sum of a power series centered at ζ, but such a power series is uniquely determined by f, so must be the given one. In particular, if f is analytic in the entire plane it can be expanded in a power series converging everywhere.

34. Suppose that f is analytic near ζ. Then

$$\frac{f(z)-f(\zeta)}{z-\zeta} \tag{7.58}$$

is also analytic near ζ. For, we can easily factor the Taylor expansion of $f(z)-f(\zeta)$. If $f(z) = \sum_{n=0}^{\infty} a_n(z-\zeta)^n$, then

$$f(z)-f(\zeta) = \sum_{n=1}^{\infty} a_n(z-\zeta)^n = (z-\zeta)\sum_{n=0}^{\infty} a_{n+1}(z-\zeta)^n$$

so (7.58) is given by $\sum_{n=0}^{\infty} a_{n+1}(z-\zeta)^n$. In particular, $z^{-1}\sin z$ is analytic on the whole plane, and has the Taylor expansion

$$z^{-1}\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(z^{n+1})!}$$

35. $\displaystyle\int_{|z|=1} \frac{\tan z\, dz}{z^2} = 2\pi i \left.\frac{\tan z}{z}\right|_{z=0} = 2\pi i$

36. $\displaystyle\int_{|z-i|=1} \frac{dz}{z^2+1} = \int_{|z-i|=1} \frac{dz}{(z+i)(z-i)} = 2\pi i \frac{1}{2i} = \pi$

37. $\displaystyle\int_{|z|=2} \frac{\sin z}{z^n}\, dz = 2\pi i (\sin z)^{(n-1)}|_{z=0}$

$$= \begin{cases} 0 & n \text{ odd} \\ \dfrac{(-1)^{n/2} 2\pi i}{(n-1)!} & n \text{ even} \end{cases}$$

38. $\displaystyle\int \frac{e^z}{(z-\zeta)^n}\, dz = 2\pi i \left.\frac{(e^z)^{(n-1)}}{(n-1)!}\right|_{z=\zeta} = \frac{2\pi i e^\zeta}{(n-1)!}$

39. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} \qquad |a| < 1$

This integral can be computed by means of Cauchy's theorem by interpreting it as an integral over the unit circle. Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

on the unit circle, we may rewrite the integral as

$$\int_{|z|=1} \left(1 - 2a\left(\frac{z + z^{-1}}{2}\right) + a^2\right)^{-1} \frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{dz}{z - az^2 - a + a^2 z}$$

$$= \frac{-1}{ia}\int_{|z|=1} \frac{dz}{z^2 - (a + (1/a))z + 1}$$

$$= \frac{-1}{ia}\int_{|z|=1} \frac{dz}{(z-a)(z-a^{-1})} \tag{7.59}$$

Since $|a| < 1$, the function $(z - a^{-1})^{-1}$ is analytic on the unit disk and the integral (7.59) can be computed by the Cauchy integral formula

$$\int_{|z|=1} \frac{dz}{(z - a^{-1})(z - a)} = 2\pi i \frac{1}{a - a^{-1}}$$

Thus

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2} - \frac{-2\pi}{a}\left(\frac{1}{a - a^{-1}}\right) = \frac{2\pi}{1 - a^2}$$

Theory of Residues

There are many definite integrals which may be computed in similar fashion. The integral formulas of complex analysis provide a powerful technique for computing such definite integrals called the **residue calculus.** We shall give a brief introduction to these methods. First, a few more illustrations

40. $$\int_0^{2\pi} \cos^6\theta\, d\theta = \int_{|z|=1} \left(\frac{z + z^{-1}}{2}\right)^6 \frac{dz}{iz}$$

$$= \frac{1}{2^6 i}\int_{|z|=1} \frac{(z^2+1)^6}{z^7}\, dz = \frac{\pi}{6!2^5}\left[(z^2+1)^6\right]^{(6)}\Big|_{z=0}$$

$$= \frac{\pi}{2^5}\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}$$

41. $$\int_0^{2\pi} \frac{d\theta}{1 + \cos^2\theta} = \int_{|z|=1} \left[1 + \left(\frac{z + z^{-1}}{2}\right)^2\right]^{-1} \frac{dz}{iz}$$

$$= \frac{4}{i}\int_{|z|=1} \frac{z\, dz}{z^4 + 6z^2 + 1} \tag{7.60}$$

We are now not in a very good position, for we cannot recognize the integrand as a Cauchy integrand. To do so we should be able to write it in the form $f(z)(z - \zeta)^{-n}$ for some function f analytic on the unit disk, and ζ in the disk. But it is not of that form. The integrand is

$$\frac{z}{(z^2 + 3 + 2\sqrt{2})(z^2 + 3 - 2\sqrt{2})}$$

$$= \frac{z}{z^2 + (3 + 2\sqrt{2})} \cdot \frac{1}{z + (-3 + 2\sqrt{2})^{1/2}} \cdot \frac{1}{z - (-3 + 2\sqrt{2})^{1/2}}$$

which has the form $f(z)(z-\alpha)^{-1}(z-\beta)^{-1}$ for *two* points α, β in the disk. However, we can still compute this integral by returning to the *proof* of Cauchy's integral formula. If Δ_1, Δ_2 are two small disks centered at α, β, respectively, then $f(z)(z-\alpha)^{-1}(z-\beta)^{-1}$ is analytic in $\Delta - (\Delta_1 \cup \Delta_2)$, so by Cauchy's theorem

$$\int_{\partial[\Delta - (\Delta_1 \cup \Delta_2)]} \frac{f(z)dz}{(z-\alpha)(z-\beta)} = 0$$

Thus, the integral (7.60) is the same as

$$\int_{\partial \Delta_1} \frac{z\,dz}{(z^2+3+2\sqrt{2})(z-\beta)(z-\alpha)} + \int_{\partial \Delta_2} \frac{z\,dz}{(z^2+3+2\sqrt{2})(z-\alpha)(z-\beta)} \tag{7.61}$$

Now these integrands are of the form $f(z)(z-\zeta)^{-1}$ with f analytic on the disk and ζ in the disk, and can be evaluated by Cauchy's integral. (7.61) is thus

$$2\pi i\left[\frac{\alpha}{(\alpha^2+3+2\sqrt{2})(\alpha-\beta)} + \frac{\beta}{(\beta^2+3+2\sqrt{2})(\beta-\alpha)}\right]$$

Since $\alpha = -(-3+2\sqrt{2})^{1/2}$, $\beta = (-3+2\sqrt{2})^{1/2}$, we obtain the result

$$\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta} = \frac{4}{i}\cdot 2\pi i\left[\frac{1}{-3+2\sqrt{2}+3+2\sqrt{2}}\right]\frac{\alpha-\beta}{\alpha-\beta} = \pi\sqrt{2}$$

The above idea of suitably generalizing the integral formula so as to accommodate a larger class of integrals is called the **residue theorem.** We shall now prove it in general.

Definition 10. Suppose that f is analytic in a neighborhood of the point ζ, except perhaps at ζ. We say that f has an **isolated singularity at** ζ. The **residue** of such a function f at ζ is defined to be

$$\operatorname{Res}(f,\zeta) = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|z-\zeta|=\varepsilon} f(z)\,dz$$

Of course, we do not a priori know that this limit exists, and therefore that the residue is well defined. However, there is no problem: for any ε and ε', we have

$$\int_{|z-\zeta|=\varepsilon} f(z)\,dz = \int_{|z-\zeta|=\varepsilon'} f(z)\,dz$$

by Cauchy's theorem, since f is analytic in the (regular) domain bounded by these two circles. Thus the limit certainly exists since it is independent of ε. Now the residue theorem says that the boundary integral of a function analytic but for isolated singularities is given by its residues; which we may calculate by the integral formula, or other available local means.

Theorem 7.11. (Residue Theorem) *Suppose that f is analytic on the regular domain D but for isolated singularities at $\zeta_1, \ldots, \zeta_n$ in D. Then*

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}(f, \zeta_i) \tag{7.62}$$

Proof. Let $\Delta_1, \ldots, \Delta_n$ be disjoint disks centered at $\zeta_1, \ldots, \zeta_n$, respectively. Then since f is analytic in $D - \cup_{i=1}^{n} D_i$, by Cauchy's theorem

$$\int_{\partial D} f(z)\,dz = \sum_{i=1}^{n} \int_{\partial \Delta_i} f(z)\,dz$$

But the sum is just (7.62) by the definition of residue.

Examples

42.

$$\int_{-\pi}^{\pi} \frac{\cos^2\theta}{1+\sin^2\theta}\,d\theta = \int_{-\pi}^{\pi} \frac{\frac{1}{4}(z+(1/z))^2}{1-\frac{1}{4}(z-(1/z))^2}\frac{dz}{iz}$$
$$= \int_{-\pi}^{\pi} \frac{-1}{iz}\frac{z^4+2z^2+1}{z^4-2z^2-3}\,dz$$

Now the roots of the denominator are

$$0 \qquad \pm\sqrt{\frac{3}{2}} \qquad \pm\frac{i}{\sqrt{2}}$$

and the integrand can be rewritten as

$$f(z) = \frac{-1}{iz} \frac{z^4 + 2z^2 + 1}{(z^2 - 3/2)(z + i/\sqrt{2})(z - (i/\sqrt{2}))}$$

The residues to be computed are those at 0, $\pm i/\sqrt{2}$. The integral around each singularity is a Cauchy integral, so we need only evaluate the relevant function at the point in question.

$$\mathrm{Res}_0\, f = \frac{1}{3i}$$

$$\mathrm{Res}_{i/\sqrt{2}}\, f = \frac{1/4 + 2(-1/2) + 1}{i(i/\sqrt{2})(-(1/2) - (3/2))(2i/\sqrt{2})} = -\frac{1}{8i}$$

$$\mathrm{Res}_{-i/\sqrt{2}}\, f = \frac{-1}{1/\sqrt{2}} \frac{1/4}{(-2)(-2i/\sqrt{2})} = \frac{1}{8i}$$

Thus, our integral is

$$2\pi i \sum \mathrm{Res}\, f = 2\pi i\left(\frac{1}{3i} - \frac{1}{8i} + \frac{1}{8i}\right) = \frac{2\pi}{3}$$

It is clear that any integral of the form

$$\int_{-\pi}^{\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where R is a quotient of polynomials, can be handled in this way by the substitutions

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad d\theta = \frac{dz}{iz}$$

The integrand then becomes a quotient of polynomials is z, and we need only compute the residues at the roots of the denominator which lie inside the unit circle. At such a root r, the integrand takes the form

$$\frac{f(z)}{g(z)(z-r)^k}$$

where f/g is analytic near r. Thus the residue is, by Cauchy's formula

$$\frac{1}{2\pi i}\int \frac{f(z)\,dz}{g(z)(z-r)^k} = \frac{1}{(k-1)!}\left(\frac{f(z)}{g(z)}\right)^{(k-1)}\Bigg|_{z=r}$$

The cases we have considered so far are those where $k = 1$. Here is an illustration of the more general case.

43.

$$\int_{-\pi}^{\pi} \frac{d\theta}{(2+\cos\theta)^2} = \int_{|z|=1} \frac{1}{[2+\frac{1}{2}(z+(1/z))]^2}\frac{dz}{iz}$$
$$= \frac{4}{i}\int_{|z|=1} \frac{z\,dz}{(z^2+4z+1)^2}$$

The roots of the denominator of the integrand are

$$-2+\sqrt{3} \qquad -2-\sqrt{3}$$

These are both double roots. We need not be concerned with the root $-2-\sqrt{3}$, since it is outside the unit disk. The integral is conveniently rewritten as

$$\frac{4}{i}\int_{|z|=1} \frac{z\,dz}{(z+2+\sqrt{3})^2(z+2-\sqrt{3})^2}$$

By Cauchy's formula the integral is evaluating the derivative of $f(z) = z(z+2+\sqrt{3})^{-2}$ at $-2+\sqrt{3}$. Now

$$f'(-2+\sqrt{3}) = -\frac{-2+\sqrt{3}-2-\sqrt{3}}{(-2+\sqrt{3}+2+\sqrt{3})^3} = \frac{1}{2\sqrt{27}}$$

Therefore, our integral is

$$\frac{4}{i}\cdot 2\pi i\cdot\frac{1}{2(27)^{1/2}} = \frac{4\pi}{(27)^{1/2}}$$

44. Occasionally, the integrand does not obligingly form itself into a Cauchy integral, and we must play around a little more

$$\int_{-\pi}^{\pi} e^{2\cos\theta}\,d\theta = \int_{|z|=1}\exp\left(z+\frac{1}{z}\right)\frac{dz}{z} = \int_{|z|=1}\frac{e^z e^{1/z}}{z}\,dz$$

The only singularity is at 0, but we cannot rearrange this in the form $f(z) \cdot z^{-1}$. Thus we must compute the integral directly by some other means. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

and

$$e^z \cdot e^{1/z} = \sum_{n=-\infty}^{\infty} \left(\sum_{\substack{i-j=n \\ i \geq 0 \\ j \geq 0}} \frac{1}{i!} \frac{1}{j!} \right) z^n$$

Thus

$$\int_{-\pi}^{\pi} e^{2\cos\theta}\, d\theta = \sum_{n=-\infty}^{\infty} \sum_{\substack{i-j=n \\ i \geq 0 \\ j \geq 0}} \frac{1}{i!j!} \int_{|z|=1} z^{n-1}\, dz$$

But that last integral is zero unless $n = 0$, in which case it is 2π. We conclude that

$$\int_{-\pi}^{\pi} e^{2\cos\theta}\, d\theta = 2\pi \sum_{\substack{i=j \\ i \geq 0}} \frac{1}{i!j!} = 2\pi \sum_{n=0}^{\infty} \frac{1}{(n!)^2}$$

Integrals from $-\infty$ to $+\infty$

The techniques of residue calculus also apply to suitable integrals of the form

$$\int_{-\infty}^{\infty} F(x)\, dx \tag{7.63}$$

If, say, $F(z)$ is analytic but for isolated singularities at $z_1, \ldots, z_k$ in the upper half plane, then

$$\int_{\partial D} F(z)\, dz = 2\pi i \sum_{i=1}^{k} \operatorname{Res}_{z_i}(F)$$

whenever D is a domain containing $z_1, \ldots, z_k$. Choosing $D = D_R = \{z\colon |z| \le R, \text{ Im } z > 0\}$, the integral is

$$\int_{-R}^{R} F(x)\,dx + \int_{H_R} F(z)\,dz$$

where H_R is the boundary in the upper half plane of the disk of radius R. Now if $F(z) \to 0$ as $|z| \to \infty$ fast enough, the integral over H_R will tend to zero and the integral from $-R$ to R will tend to (7.63). We shall say that F is **dissipative** in the upper half plane in this case. Thus we conclude that when F is dissipative in the half plane Π,

$$\int_{-\infty}^{\infty} F(x)\,dx = 2\pi i \sum_{z \in \Pi} \text{Res}_z(f)$$

45.

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1} = \lim_{R\to\infty} \int_{\partial D_R} \frac{z^2\,dz}{z^4+1}$$

For

$$\left|\int_{H_R} \frac{z\,dz}{z^4+1}\right| = \left|\int_0^{\pi} \frac{R^2 e^{2i\theta}\cdot Re^{i\theta} i\,d\theta}{R^4 e^{Ri\theta}+1}\right|$$
$$\le \frac{2\pi R^3}{R^4-1} \to 0 \qquad \text{as } R \to \infty$$

Now the roots of $z^4 + 1$ are $(\pm 1 \pm i)/\sqrt{2}$. Those in the upper half plane are $a = (1+i)/\sqrt{2}$, $b = (1-i)/\sqrt{2}$. Thus

$$\text{Res}_a\left(\frac{z}{z^4+1}\right) = \frac{\left[\dfrac{1+i}{\sqrt{2}}\right]\left[\dfrac{1+i}{\sqrt{2}}\right]}{\left[\dfrac{1+i}{\sqrt{2}} - \dfrac{-1-i}{\sqrt{2}}\right]\left[\dfrac{1+i}{\sqrt{2}} - \dfrac{-1+i}{\sqrt{2}}\right]\left[\dfrac{1+i}{\sqrt{2}} - \dfrac{1-i}{\sqrt{2}}\right]}$$
$$= \frac{1+i}{8i\sqrt{2}}$$

$$\text{Res}_b\left(\frac{z}{z^4+1}\right) = \frac{1-i}{8i\sqrt{2}}$$

Finally,

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1} = 2\pi i\left[\frac{1+i}{8i\sqrt{2}} + \frac{1-i}{8i\sqrt{2}}\right] = \frac{\pi}{2\sqrt{2}}$$

A condition on F that guarantees that it is dissipative is that F is the quotient of two polynomials such that the denominator is of degree two more than the numerator (see Problem 37).

46. Compute

$$\int_{-\infty}^{\infty} \frac{e^{-iax}\,dx}{1+x^2} \qquad a > 0 \tag{7.64}$$

Now, we would hope to apply the residue theorem to $e^{-iz}(1+z^2)^{-1}$. For $z = x + iy$, this becomes

$$\frac{e^{y}\,e^{-iax}}{1+(x+iy)^2}$$

which is hardly dissipative for $y > 0$. But it is dissipative in the lower half plane:

$$\left|\int_{\substack{|z|=R\\ y<0}} \frac{e^{-iaz}\,dz}{1+z^2}\right| \le \left|\int_{-\pi}^{0} \frac{\exp[-ia(R\cos\theta + i\sin\theta)]Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}\right|$$

$$\le \frac{R}{R^2-1}\int_{-\pi}^{0} \exp(-aR\sin\theta)\,d\theta \le \frac{\pi}{R^2-1} \to 0$$

as $R \to \infty$. Thus we compute (7.64) by residues over the lower half plane:

$$\int_{-\infty}^{\infty} \frac{e^{-iax}\,dx}{1+x^2} = -2\pi i\,\mathrm{Res}_{-i}\left(\frac{e^{-iaz}}{1+z^2}\right) = 2\pi i\,\frac{e^{-a}}{-2i} = \frac{\pi}{e^{a}}$$

(The sign changes since the x axis is oriented opposite to the orientation it obtains as boundary of the lower half plane.) Notice, by the way, that

$$\int_{-\infty}^{\infty} \frac{\cos ax\,dx}{1+x^2} = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{-iax}\,dx}{1+x^2} = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{iax}\,dx}{1+x^2} = \frac{\pi}{e^{a}}$$

Since

$$\int_{-\infty}^{\infty} \frac{\sin ax\,dx}{1+x^2} = 0$$

(the integrand is an odd function), we obtain

$$\int_{-\infty}^{\infty} \frac{e^{iax}\,dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{e^{-iax}}{1+x^2}\,dx = \frac{\pi}{e^a} \qquad a > 0$$

● EXERCISES

19. Perform the indicated integrations by residues:

(a) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{\cos^2\theta + 2\sin^2\theta}$

(b) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{(\cos^2\theta + 2\sin^2\theta)^2}$

(c) $\displaystyle\int_{|z|=2} \frac{e^{iz}}{z(z-1)^4}\,dz$

(d) $\displaystyle\int_{|z|=1} \frac{e^z z\,dz}{(4z^2+1)}$

(e) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}$

(f) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\alpha\sin\theta}, \; \alpha < 1$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{1+x^4}$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^6+1}$

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)}\,dx$

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+3x+2}$

(l) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^{10}}$

20. Suppose that f is analytic in a neighborhood of ζ, and

$$f^{(j)}(\zeta)=0 \qquad 0 \leq j < k$$

Show that

$$g(z)=\frac{f(z)-f(\zeta)}{(z-\zeta)^k}$$

is an analytic function.

21. Suppose that f is analytic in a disk centered at ζ, and all derivatives of f vanish at ζ. Then f is identically zero.

22. Suppose that f is analytic in the punctured disk $0 < |z-\zeta| < R$ and bounded. Then, defining f at ζ by

$$f(\zeta)=\lim_{z\to\zeta} f(z)$$

the extended function is analytic.

• PROBLEMS

36. If $\{f_n\}$ is a convergent sequence of analytic functions in the domain D, then the limit function is also analytic.

37. If

$$F(z)=\frac{P(z)}{Q(z)}$$

where P, Q are polynomials, then F is dissipative if the degree of Q is 2 more than that of P.

38. Suppose that f is analytic in the punctured disk $0 < |z-\zeta_0| \leq R$.

(a) Show that

$$\int_{|z|=r} \frac{f(z)}{(z-\zeta_0)^n}\,dz \qquad 0<r<R$$

is independent of r.

(b) Fix some $r_0 < R$. Show that if $r_0 < |\zeta - \zeta_0| < R$,

$$f(\zeta) = \frac{1}{2\pi i} \int_{|\zeta-\zeta_0|=R} \frac{f(z)\,dz}{z-\zeta} - \frac{1}{2\pi i} \int_{|\zeta-\zeta_0|=r_0} \frac{f(z)\,dz}{z-\zeta}$$

(c) Expand f in a series of the form

$$f(\zeta) = \sum_{n=-\infty}^{\infty} a_n(\zeta-\zeta_0)^n \tag{7.65}$$

called the **Laurent expansion of** f, by noticing that

$$\frac{1}{z-\zeta} = \frac{1}{z-\zeta_0}\left[1-\left(\frac{\zeta-\zeta_0}{z-\zeta_0}\right)\right]^{-1} = \sum_{n=0}^{\infty} \frac{(\zeta-\zeta_0)^n}{(z-\zeta_0)^{n+1}}$$

for $|\zeta - \zeta_0| = R$, $|z - \zeta_0| < R$, and

$$\frac{1}{z-\zeta} = \sum_{n=0}^{\infty} \frac{(z-\zeta_0)^n}{(\zeta-\zeta_0)^{n+1}}$$

for $|z - \zeta_0| = r$, and $|\zeta - \zeta_0| > r$.

(d) Show that $\operatorname{Res}_\zeta f = a_{-1}$.

39. Equation (7.65) can be verified in another way. Expand f in a Fourier series around each circle $|z - \zeta_0| = r$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\theta} \qquad z = re^{i\theta} \tag{7.66}$$

(a) The Cauchy–Riemann equations imply that

$$r\frac{\partial f}{\partial r} + i\frac{\partial f}{\partial \theta} = 0$$

(b) Differentiating (7.66), we obtain

$$0 = \sum_{n=-\infty}^{\infty} (ra_n' - na_n)e^{in\theta}$$

Conclude that $a_n(r) = A_n r^n$. Thus (7.66) becomes

$$f(z) = \sum_{n=-\infty}^{\infty} A_n r^n e^{in\theta} = A_0 + \sum_{n=1}^{\infty}(A_{-n}z^{-n} + A_n z^n)$$

40. Suppose that f is one-to-one in the domain D. Then by the residue theorem

$$\frac{1}{2\pi i}\int_{\partial D}\frac{z\,dz}{w-f(z)}=0$$

if w is not a value of f in D. Suppose $f(a)=w$. Then

$$a=f^{-1}(w)=\frac{1}{2\pi i}\int_{\partial D}\frac{z\,dz}{w-f(z)}$$

Conclude that the inverse of a one-to-one analytic function is again analytic.

7.8 Summary

Let $\mathbf{p} \in R^n$ and suppose $\mathbf{f}$ is an R^m-valued function defined in a neighborhood of $\mathbf{p}$. f is *differentiable at* $\mathbf{p}$ if there is a linear transformation $T: R^n \to R^m$ such that

$$\frac{\|\mathbf{f}(\mathbf{p}+\mathbf{v})-\mathbf{f}(\mathbf{p})-T(\mathbf{v})\|}{\|\mathbf{v}\|}\to 0 \text{ as } \mathbf{v}\to \mathbf{0}$$

T is called the *differential of* $\mathbf{f}$ at $\mathbf{p}$ and is denoted $d\mathbf{f}(\mathbf{p})$.

The differential is linear in the function $\mathbf{f}$ and also satisfies

$$d\langle \mathbf{f}, \mathbf{g}\rangle = \langle d\mathbf{f}, \mathbf{g}\rangle + \langle \mathbf{f}, d\mathbf{g}\rangle$$

Let U be a domain in R^n. A *system of coordinates* on U is an n-tuple of C^1 functions $\mathbf{y}$ such that

(i) if $\mathbf{p} \neq \mathbf{q}$, $\mathbf{y}(\mathbf{p}) \neq \mathbf{y}(\mathbf{q})$
(ii) $d\mathbf{y}(\mathbf{p})$ is nonsingular at all $\mathbf{p} \in U$

The matrix

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} = \frac{\partial y^i}{\partial x^j}$$

is called the *Jacobian of the coordinate change.*

THE CHAIN RULE. The differentials of composed mappings compose as linear transformations:

$$d(\mathbf{g} \circ \mathbf{f})(\mathbf{p}) = d\mathbf{g}(\mathbf{f}(\mathbf{p})) \circ d\mathbf{f}(\mathbf{p})$$

INVERSE MAPPING THEOREM. Suppose $\mathbf{F}$ is a C^1 R^n-valued function defined in a neighborhood of $\mathbf{p}_0$ such that $d\mathbf{F}(\mathbf{p}_0)$ is nonsingular. Then there are neighborhoods N of $\mathbf{p}_0$ and U of $\mathbf{F}(\mathbf{p}_0)$ and a C^1 mapping $G: U \to N$ such that $\mathbf{G} = \mathbf{F}^{-1}$.

Let D be a domain in R^n. A *differential form* on D is a function which associates to each point $\mathbf{p}$ in D a linear function $\omega(\mathbf{p})$ on R^n. A differential form has the form

$$\omega(\mathbf{p}) = \sum_{i=1}^{n} a_i(\mathbf{p})\, dx^i(\mathbf{p})$$

ω is said to be C^k on D if all the functions $a_1, \ldots, a_n$ are C^k. If ω is the differential of a function we must have

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i} \qquad 1 \leq i, j \leq n \tag{7.67}$$

A differential form is *exact* if it is the differential of a function, and *closed* if (7.67) holds.

Suppose that F is a force field defined in a domain D in R^n, and Γ is an oriented path defined in D. The *work required to move a unit mass along* Γ is

$$W(\Gamma, \mathbf{F}) = -\int_a^b \langle \mathbf{F}(t), \mathbf{g}'(t) \rangle\, dt$$

where g furnishes a parametrization of Γ.

A field is *conservative* if $W(\Gamma, \mathbf{F}) = 0$ over all closed paths Γ. A potential function for a field $\mathbf{F}$ is a real-valued function Π such that

$$W(\Gamma, \mathbf{F}) + \Pi(\mathbf{p}') - \Pi(\mathbf{p})$$

is the same for every oriented path Γ from $\mathbf{p}$ to $\mathbf{p}'$.

Suppose D is a domain such that any two points can be joined by a path in D. Then

(i) every field, conservative in D, has a potential function
(ii) two potentials of a given field differ by a constant
(iii) If $\mathbf{F} = (f_1, \ldots, f_n)$ has the potential Π, $d\Pi = \sum f_i\, dx^i$

LINE INTEGRAL OF A DIFFERENTIAL FORM. Let Γ be an oriented path in a domain on which the form ω is defined. If $\Gamma = \sum_{i=1}^{s} \Gamma_i$, define

$$\int_\Gamma \omega = \sum_{i=1}^{s} \int_{a_i}^{b_i} \omega(\mathbf{g}_i(t))(\mathbf{g}_i'(t))\, dt$$

If $\mathbf{T}$ is the tangent to Γ,

$$\int_\Gamma \omega = \int_\Gamma \omega(\mathbf{T})\, ds$$

Let $\omega = \sum a_i\, dx^i$ be a C^1 differential form defined on D. $\omega = df$ for some function f

(i) if and only if the field $(a_1, \ldots, a_n)$ is conservative
(ii) if and only if $\oint_\Gamma \omega = 0$ for all closed curves
(iii) only if

$$\frac{\partial a_i}{\partial x^j} = \frac{\partial a_j}{\partial x^i} \qquad \text{for all } i, j$$

throughout D.

POINCARÉ'S LEMMA. Suppose that D is a domain such that for some fixed point $\mathbf{p}_0$ in D and every $\mathbf{p} \in D$, the line segment joining $\mathbf{p}_0$ to $\mathbf{p}$ is contained in D. Then every closed form is exact in D.

In two dimensions a differential form has the form $\omega = p\, dx + q\, dy$. If ω is C^1 we shall denote the function

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

by $d\omega$. A *regular domain* in R^2 is bounded by a piecewise C^1 curve. We orient this curve so that its principal normal points into D (it winds counterclockwise around D). When so oriented we shall denote the bounding path by ∂D.

GREEN'S THEOREM. If ω is a C^1 differential form defined on the regular domain D,

$$\int_{\partial D} \omega = \int_D d\omega$$

Integration under a coordinate change. Suppose

$$x = x(u, v)$$
$$y = y(u, v)$$

is a coordinate change on the domain D in R^2. Let E be the domain in the uv plane corresponding to D. If F is continuous on D, then

$$\int_D f = \int_E f(x(u, v), y(u, v)) \left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Let $\mathbf{v} = (v, w)$ be a C^1 vector field. The *divergence* of $\mathbf{v}$ is

$$\operatorname{div} \mathbf{v} = \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}$$

DIVERGENCE THEOREM. If $\mathbf{v}$ is a C^1 vector field defined on the regular domain D,

$$\int_{\partial D} \langle \mathbf{v}, \mathbf{N} \rangle\, ds = \int_D \operatorname{div} \mathbf{v}$$

A C^2 function f is the potential of a conservative divergence-free flow if and only if it is harmonic.

CAUCHY'S THEOREM. If f is a C^1 complex differentiable function defined on the regular domain D, then

$$\int_D f\, dz = 0$$

CAUCHY INTEGRAL FORMULA. Under the same hypotheses on f, if $\zeta \in D$,

$$f(\zeta) = \frac{1}{2\pi i} \int_D \frac{f(z)\, dz}{z - \zeta}$$

MAXIMUM PRINCIPLE. If f is analytic on D, it attains its maximum on ∂D.

Theorem. Let f be a C^1 complex-valued function defined on the regular domain D. The following assertions are equivalent

(i) for any $\zeta \in D$, and some R such that $\Delta(\zeta, R) \subset D$ f is the sum in $\Delta(\zeta, R)$

of a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \zeta)^n \tag{7.68}$$

(ii) replace the word *some* in (i) by *any*
(iii) f is complex differentiable
(iv) f satisfies the Cauchy–Riemann equations

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}$$

(v) $f\,dz$ is closed
(vi) for any $\zeta \in D$

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\,dz}{z - \zeta}$$

In case f has these properties (f is *analytic*), the coefficients a_n of (7.68) are given by

$$a_n = \frac{f^{(n)}(\zeta)}{n!} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\,dz}{(z - \zeta)^{n+1}}$$

If f is analytic in $\{0 < |z - z_0| < R\}$, we say that f has an isolated singularity at z_0. In this case the integrals

$$\frac{1}{2\pi i} \int_{|z - z_0| = r} f(z)\,dz$$

are all the same for $0 < r < R$. Their common value is the *residue* of f at z_0, denoted Res (f, z_0).

RESIDUE THEOREM. If f is an analytic function on the regular domain D, except for isolated singularities at $z_1, \ldots, z_n$ in D, then

$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}(f, z_i)$$

● FURTHER READING

The general theorems on differentiation in R^n are fully discussed in:

H. K. Nickerson, N. Steenrod, D. C. Spencer, *Advanced Calculus*, D. Van Nostrand Company, Inc., Princeton, N. J., 1957.

M. E. Munroe, *Modern Multidimensional Calculus*, Addison-Wesley, Reading, Mass., 1963.

L. Loomis and S. Sternberg, *Advanced Calculus*, Addison-Wesley, Reading, Mass., 1968.

For further information on complex analytic functions see

Z. Nehari, *Introduction to Complex Analysis*, Allyn and Bacon, Inc., Boston, 1961.

H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Addison-Wesley, Reading, Mass., 1963.

E. Hille, *Analytic Function Theory*, Ginn and Company, Boston, 1959.

L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1953.

● MISCELLANEOUS PROBLEMS

41. Prove the assertion concerning integration under a coordinate change as given in the summary (where no reference to the orientation is made).

42. Show that if ω is a differential form of compact support in R^2, that

$$\int_{R^2} d\omega = 0$$

43. Recall the definition of connectedness given in Problem 78 of Chapter 2. Show that a domain in R^2 is connected if and only if it is pathwise connected.

44. If $\omega = p\,dx + q\,dy$ is a C^1 form, define

$$*\omega = -q\,dx + p\,dy$$

(a) Show that for any regular domain D,

$$\int_{\partial D} \omega(\mathbf{N})\,ds = \int_D d{*\omega}$$

where $\mathbf{N}$ is the interior normal to D.

(b) Show that the function u is harmonic if and only if $d{*du} = 0$.

(c) ω is (locally) the differential of a harmonic function if and only if $d\omega = 0$, $d{*\omega} = 0$.

45. If u is a harmonic function in the domain D and if $*du$ is exact in D, then u is the real part of an analytic function in D.

46. If u is harmonic in D, and Γ is a closed path in D, the integral

$$\frac{1}{2\pi}\int_{\Gamma} {}^{*}du$$

is called the *period of u about* Γ. Show that u has zero periods about all paths if and only if u is the real part of an analytic function. Show that $\exp(u)$ is the modulus of an analytic function if and only if u has integer periods.

47. Let $D = R^2 - \{\mathbf{p}_1, \ldots, \mathbf{p}_s\}$, where $\mathbf{p}_1, \ldots, \mathbf{p}_s$ are s distinct points in the plane. Show that there is an s-dimensional space L of harmonic functions which are *not* the real part of an analytic function in D such that every harmonic function has the form $u = u_1 + \operatorname{Re} f$, $\mathbf{u}_1 \in L$, f analytic in D. (Recall Problem 33.)

48. **The Gamma function.** Define

$$\Gamma(z) = \int_0^\infty \exp[(z-1)\ln t - t]\,dt = \int_0^\infty t^{z-1}e^{-t}\,dt$$

(a) Show that $\Gamma(n) = n!$

(b) Show by integration by parts that

$$\Gamma(z+1) = z\Gamma(z)$$

(c) Show that Γ is an analytic function in the half plane $\{\operatorname{Re} z > 1\}$ (differentiate under the integral sign).

49. (a) Show that for any $a > 0$ the function

$$\Gamma_a(z) = \int_a^\infty t^{z-1}e^{-t}\,dt$$

is analytic on the entire plane.

(b) Substitute

$$e^{-t} = \sum (-1)^n \frac{t^n}{n!}$$

into the integral

$$\int_0^1 t^{z-1}e^{-t}\,dt$$

to obtain the formula

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} + \Gamma_1(z)$$

Justify that substitution.

(c) If $\operatorname{Re} z > 1$, does $\lim \Gamma_a(z) = \Gamma(z)$ as $a \to 0$?

(d) Use the result of part (b) to extend Γ to a function analytic on the entire plane, but for isolated singularities at $0, -1, -2, \ldots$.

(e) Calculate the residue of Γ at those points.

50. Find the residue at the origin of

$$\exp\left(z + \frac{1}{z}\right)$$

51. Compute the Fourier transform of $(1 + x^2)^{-1}$: find

$$\hat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2}\, dx$$

(use Example 46).

52. Compute the Fourier transforms of these functions:

(a) $(1 + x^4)^{-1}$.

(b) $(1 + x^2)^{-1}(a^2 + x^2)^{-1}$.

(c) $\dfrac{x^2}{(1+x^2)^2}$.

(d) $\dfrac{\cos x}{(1+x^2)}$.

53. Suppose $\{f_n\}$ is a sequence of analytic functions in D, and $\lim f_n = f$ uniformly in D. Show that f is analytic.

54. Prove: If f is C^1 in D and $\int_{\partial\Delta} f\,dz = 0$ for all disks Δ contained in D, then f is analytic.

55. Morera's theorem. Suppose f is a continuous complex-valued function defined in D such that

$$\int_\Gamma f\,dz = 0$$

over every closed path Γ in D. Then f is analytic. (*Hint*: Let F be a potential function for $f\,dz$ and show that F is complex differentiable.)

56. If $f = u + iv$ is an analytic function in the domain D, then u is the potential of a divergence-free velocity field. Show that the curves $\{v = \text{constant}\}$ are the path lines of the associated flow.

57. Let f be analytic in the domain $\{0 < |z - z_0| < R\}$ f is said to be **meromorphic** at z_0 if there is a function g analytic in a neighborhood of z_0 such that $f \cdot g$ extends analytically across z_0. Verify that these are equivalent conditions for meromorphicity.

(a) the Laurent expansion (7.65) of f about z_0 has only finitely many negative terms.

(b) there is an n such that $(z - z_0)^n f$ extends analytically across z_0.

58. Show that if f is analytic in the domain D except for isolated singularities at $p_1, \ldots, p_s$, where it is meromorphic, then there is a polynomial P such that $f \cdot P$ extends analytically to all of D.

59. If f is meromorphic at z_0, is $\exp(f)$ also meromorphic there?

60. **Schwarz's lemma.** Suppose that f is analytic on the disk $\{z \in C: |z| \leq 1\}$, and

(i) $\max\{|f(z)|: |z| = 1\} = M$

(ii) $f(0) = 0$

Show that for any z in that disk

$$|f(z)| \leq M|z|$$

(*Hint*: Apply the maximum principle to $z^{-1}f$.)

61. Under the same hypotheses as above show that

$$|f'(0)| \leq 1$$

and if $|f'(0)| = 1$, then $f(z) = cz$ for some constant c of modulus 1.

62. Let f be in $S(R)$, and suppose that $f(t) = 0$ for negative t. Show that

$$\hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{itz}\, dt$$

is an analytic function for z in the upper half plane. Notice that $\hat{f}(iy) = (2\pi)^{1/2}L(f)$.

63. Suppose that f is analytic and dissipative in the upper half plane and f is in $S(R)$ on the real axis. Show that there is a function $g \in S(R)$ with $g(t) = 0$ for negative t such that $f(z) = \hat{g}(z)$. (*Hint*: Let

$$\hat{g}(t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} f(x)e^{-itx}\, dt\right)$$

Then, by Fourier inversion, g and f are analytic in the upper half plane and have the same values on the real axis. Verify that $g(t) = 0$ for negative t by Cauchy's theorem.)